ALMOST CONTRA-I-CONTINUOUS MULTIFUNCTIONS

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Abstract: In this paper, we introduce and study the concept of almost contra-I-continuous multifunctions on ideal topological spaces.

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1. INTRODUCTION

One of the important and basic topics in the theory of classical point set topology and several branches of Mathematics, which have been researched by many authors, is continuity of functions. Various types of continuous functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good many of them have been extended to the setting of multifunction. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [6] and Vaidyanathaswamy, [11]. An ideal I on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star : \mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [11] of A with respect to τ and I, is defined as follows: for $A \subset X$, $A^\star(\tau, I) = \{x \in X | U \cap A \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \,|\, x \in U\}$. A Kuratowski closure operator $Cl^\star(.)$ for a topology $\tau^\star(\tau, I)$ called the *-topology, finer than τ is defined by $Cl^\star(A) = A \cup A^\star(\tau, I)$ when there is no chance of confusion, $A^\star(I)$ is denoted by $A^\star$. If I is an ideal on X, then (X, τ, I) is called an ideal topological space. Akdag [2] introduced and studied the concept of I-continuous multifunctions on ideal topological spaces. In this paper, we define contra-I-continuous multifunctions and obtain some characterizations and some basic properties of such multifunctions.

2. PRELIMINARIES

Let A be a subset of a topological space (X, τ). For a subset A of (X, τ), Cl(A) and Int(A) denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset S of an ideal topological space (X, τ, I) is I-open [1] if $S \subset Int(S^*)$. The complement of an I-open set is called and I-closed set. The intersection of all I-closed sets containing S is called the I-closure of S and is denoted by ICl(S). The family of all I-open (resp. I-closed) sets of (X, τ, I) is denoted by IO(X) (resp. IC(X)). The family of all I-open (resp. I-closed) sets of (X, τ, I) containing a point $x \in X$ is denoted by IO(X, x) (resp. IC(X, x)). By a multifunction $F : X \to Y$, following [3], we shall denote the upper and lower inverse of a set

B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(Y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$ and for each $A \subset X$, $F(A) = \cup_{x \in A} F(x)$. Then F is said to be surjection if F(x) = y and injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

Definition 2.1. A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be[2]:

(1) upper I-continuous if for each point $x \in X$ and each open set V containing F(x), there exists $U \in IO(X, x)$ such that $F(U) \subset V$;

(2) lower I-continuous if for each point $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in IO(X, x)$ such that $U \subset F^-(V)$.

III. ALMOST CONTRA-I-CONTINUOUS MULTIFUNCTIONS

Definition 3.1. A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be:

(1) upper almost contra-I-continuous if for each point $x \in X$ and each closed set V containing F(x), there exists $U \in IO(X, x)$ such that $F(U) \subset int(cl(V))$;

(2) lower almost contra-I-continuous if for each point $x \in X$ and each closed set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in IO(X, x)$ such that $U \subset F^- (int(cl(V))$.

The following examples show that the concepts of upper I-continuity (resp. lower I-continuity) and upper almost contra-I -continuity (resp. lower almost contra-I-continuity) are independent of each other.

Example 3.2. Let X = {a, b, c}, $\tau = \{\emptyset, X\}$, $\sigma = \{\emptyset, \{b, c\}, X\}$ and $I = \{\emptyset\}$. The multifunction F: $(X, \tau, I) \to (X, \sigma)$ defined by F(x) ={x} for all $x \in X$ is upper I-continuous but is not upper almost contra-I-continuous.

Example 3.3. Let X = {a, b, c}, $\tau = \{\emptyset, X\}$, $\sigma = \{\emptyset, \{a\}, X\}$ and $I = \{\emptyset\}$. The multifunction F: $(X, \tau, I) \to (X, \sigma)$ defined by F(x) ={x} for all $x \in X$ is upper almost contra-I-continuous but is not upper I -continuous.

Theorem 3.4. The following statements are equivalent for a multifunction $F : (X, \tau, I) \to (Y, \sigma)$:

(1) F is upper almost contra-I-continuous;

(2) $F^+(V) \in IO(X)$ for every regular closed subset V of Y ;

(3) $F^-(V) \in IC(X)$ for every regular open subset V of Y ;

(4) for each $x \in X$ and each regular closed set K containing F(x), there exists $U \in IO(X, x)$ such that if $y \in U$, then $F(y) \subset K$.

Proof. (1) $\Leftrightarrow$ (2): Let V be a regular closed subset in Y and $x \in F^+(V)$. Since F is upper almost contra-I-continuous, there exists $U \in IO(X, x)$ such that $F(U) \subset V$. Hence, $F^+(V)$ is I-open in X. The converse is similar.

(2) $\Leftrightarrow$ (3): It follows from the fact that $F^+(Y \setminus V) = X \setminus F^-(V)$ for every subset V of Y.

(3) $\Leftrightarrow$ (4): This is obvious.

Theorem 3.5. The following statements are equivalent for a multifunction $F : (X, \tau, I) \rightarrow (Y,\sigma)$

(1) F is lower almost contra-I-continuous;

(2) $F^-(V) \in IO(X)$ for every regular closed subset V of Y ;

(3) $F^+(K) \in IC(X)$ for every regular open subset K of Y ;

(4) for each $x \in X$ and each regular closed set K such that $F(x) \cap K \neq \emptyset$, there exists $U \in IO(X, x)$ such that if $y \in U$, then $F(y) \subset K \neq \emptyset$.

Proof. The proof is similar to that of Theorem 3.4.

Definition 3.6. A topological space (X,τ) is said to be semi-regular [10] if for each open set U of X and for each point $x \in U$, there exists a regular open set V such that $x \in V \subset U$.

Definition 3.7. [12] Let (X,τ) be a topological space and A a subset of X and x a point of X. Then

(1) x is called δ-cluster point of A if $A \cap Int(Cl(U)) \neq \emptyset$, for each open set U containing x.

(2) the family of all δ-cluster point of A is called the δ -closure of A and is denoted by $Cl\,\delta\,(A)$.

(3) A is said to be δ-closed if $Cl\,\delta\,(A) = A$. The complement of a δ-closed set is said to be a δ-open set.

Theorem 3.8. For a multifunction $F : (X, \tau, I) \rightarrow (Y,\sigma)$, where Y is semi-regular, the following are equivalent:

(1) F is upper almost contra-I-continuous;

(2) $F^+(cl(int((B))) \in IO(X)$ for every subset B of Y;

(3) $F^+(K) \in I\,O(X)$ for every δ-closed subset K of Y;

(4) $F^-(V) \in I\,C(X)$ for every δ-open subset V of Y.

Proof. (1) $\Longrightarrow$ (2): Let B be any subset of Y. Then cl(int((B)) is regular closed and by Theorem 3.4, $F^+(cl(int((B)))\in IO(X)$.

(2) $\Longrightarrow$ (3): Let K be a δ-closed set of Y . Then $Cl\,\delta\,(K) = K$. By (2), $F^+(K)$ is I-open.

(3) $\Longrightarrow$ (4): Let V be a δ-open set of Y . Then $Y \setminus V$ is δ-closed. By (3), $F^+(Y \setminus V) = X \setminus F^-(V)$ is I -open. Hence, $F^-(V)$ is I-closed.

(4) $\Longrightarrow$ (1): Let V be any open set of Y . Since Y is semi-regular, V is δ-open. By (4), $F^-(V)$ is I-closed and by Theorem 3.4, F is upper almost contra-I-continuous.

Theorem 3.9. For a multifunction $F : (X, \tau, I) \rightarrow (Y,\sigma)$, where Y is semi-regular, the following are equivalent:

(1) F is lower almost contra-I-continuous;

(2) $F^-(Cl\,\delta\,(B)) \in IO(X)$ for every subset B of Y;

(3) $F^-(K) \in IO(X)$ for every δ-closed subset K of Y;

(4) $F^+(V) \in IC(X)$ for every δ-open subset V of Y .

Proof. The proof is similar to that of Theorem 3.8.

Definition 3.10. [8] A subset A of an ideal topological space (X, τ, I) is said to be I-compact relative to X if for every cover {Uα : $\alpha \in \triangle$} of A by open subsets of X, there exists a finite subset △0 of △ such that A\ ∪ {Uα : $\alpha \in \triangle 0$} ∈ I.

Definition 3.11. A subset A of an ideal topological space (X, τ, I) is said to be I-closed relative to X if for every cover {Uα : $\alpha \in \triangle$} of A by closed subsets of X, there exists a finite subset △0 of △ such that A\ ∪ {Uα : $\alpha \in \triangle 0$} ∈ I.

Theorem 3.12. Let F : (X, τ, I) → (Y,σ) be an upper almost contra-I-continuous surjective multifunction and F(x) is strongly S-closed relative to Y for each x ∈ X. If A is I-compact relative to X, then F(A) is F(I)-closed relative to Y .

Proof. Let {V_i : i ∈ △} be any cover of F(A) by closed sets of Y . For each x ∈ A, there exists a finite subset △(x) of △ such that F(x) ⊂∪ {V_i : i ∈ △ (x)}. Put V (x) = ∪ {V_i : i ∈ △ (x)}. Then F(x) ⊂ V (x) and there exists U(x) ∈ IO(X, x) such that F(U(x)) ⊂ V (x). Since {U(x) : x ∈ A} is a cover of A by I-open sets in X, there exists a finite number of points of A, say, x_1, x_2,....x_n such that A\∪ {U(x_i) : 1 = 1, 2,….n} ∈ I. Therefore, we obtain F(A)\ $\cup_{i=1}^{n} \cup_{i=\triangle(xi)}$ Vi ∈ F(I). This shows that F(A) is I-closed relative to Y .

Corollary 3.13. Let F : (X, τ, I) → (Y,σ) be an upper almost contra-I-continuous surjective multifunction and F(x) is I-compact relative to Y for each x ∈ X. If X is I-compact, then Y is F(I)-closed.

Definition 3.14. [4] Let A and B be subsets of an ideal topological space (X, τ, I) such that A ⊂ B ⊂ X. Then (B, τ|B, I |B) is an ideal topological space with an ideal I |B = {I ∈ I |I ⊂ B} = {I ∩ B|I ∈ I }.

Lemma 3.15. Let A and B be subsets of an ideal topological space (X, τ, I).

(1) If A ∈ I O(X) and B ∈ τ , then A ∩ B ∈ I O(B) [5];

(2) If A ∈ I O(B) and B ∈ I O(X), then A ∈ I O(X) [7].

Theorem 3.16. Let F: (X, τ, I) → (Y,σ) be a multifunction and U an open subset of X. If F is a upper almost contra-I-continuous (resp. lower almost contra-I-continuous), then $F_{|U}$: U → Y is an upper almost contra-I-continuous (resp. lower almost contra-I-continuous) multifunction.

Proof. Let V be any closed set of Y . Let x ∈ U and x ∈ $F_{|U}^{-}$(V). Since F is lower almost contra-I-continuous multifunction, there exists a I-open set G containing x such that G ⊂ F^-(V). Then x ∈ G∩ U ∈ I O(A) and G ∩ U ⊂ $F_{|U}^{-}$ (V) . This shows that $F_{|U}$ is a lower almost contra-I-continuous. The proof of the upper almost contra-I-continuous of F|U is similar.

Theorem 3.17. Let $\{U_i : i \in \triangle\}$ be an open cover of a topological space X. A multifunction $F : (X, \tau, I) \rightarrow (Y,\sigma)$ is upper almost contra-I-continuous if and only if the restriction $F_{|U} : U_i \rightarrow Y$ is upper almost contra- I -continuous for each $i \in \triangle$.

Proof. Suppose that F is upper almost contra-I-continuous. Let $i \in \triangle$and $x \in U_i$ and V be a closed set of Y containing $F_{|Ui}(x)$. Since F is upper almost contra-I-continuous and $F(x) = F_{|Ui}(x)$, there exists $G \in I\,O(X, x)$ such that $F(G) \subset V$. Set $U = G \cap U_i$, then $x \in U \in IO(U_i, x)$ and $F_{|Ui}(U) = F(U) \subset V$. Therefore, $F_{|Ui}$ is upper almost contra-I-continuous.

Conversely, let $x \in X$ and V be a closed subset of Y containing F(x).

There exists $i \in \triangle$ such that $x \in Ui$. Since $F_{|Ui}$ is upper almost contra-I- continuous and $F(x) = F_{|Ui}(x)$, there exists $U \in I\,O(U_i, x)$ such that $F_{|Ui}(U) \subset V$. Then we have $U \in IO(X, x)$ and $F(U) \subset V$. Therefore, F is upper almost contra-I-continuous.

Theorem 3.18. Let X and X_j be topological spaces for $i \in I$. If a multifunction $F : X \rightarrow \prod_{i\in I} X_i$ is an upper (lower) contra-I-continuous multifunction, then $P_i o F$ is an upper (lower) contra-I -continuous multifunction for each $i \in I$,where $Pi : \prod_{i\in I} X_i \rightarrow X_i$ is the projection for each $i \in I$.

Proof. Let H_i be a closed subset of X_i. We have $(P_i\ o\ F)^+(H_i) = F^+(P_i^{\ +}(H_i)) = F+(H_i \times \prod_{i\neq j} X_j)$. Since F an upper almost contra-I-continuous multifunction, $F^+(H_i \times \prod_{i\neq j} X_j)$ is I-open in X. Hence,$P_i\ o\ F$ is an upper (lower) contra-I-continuous.

Definition 3.19. [9] A topological space (X, τ) is said to be ultranormal if each pair of nonempty disjoint closed sets can be separated by disjoint clopen sets.

Definition 3.20. [7] An ideal topological space (X, τ, I) is said to be I-normal if each pair of nonempty disjoint closed sets can be separated by disjoint I-open sets.

Theorem 3.21. If $F : (X, \tau, I) \rightarrow (Y,\sigma)$ is an upper almost contra-I-continuous punctually closed multifunction and (Y, σ) is ultranormal, then (X, τ, I) is I-normal.

Proof. The proof follows from the respective definitions.

Definition 3.22. Let A be a subset of an ideal topological space (X, τ, I). The I-frontier of A denoted by $IF_r(A)$, is defined as follows: $IF_r(A) = I\,Cl(A) \cap I\,Cl(X\backslash A)$.

Theorem 3.23. The set of points x of X at which a multifunction $F : (X, \tau, I) \rightarrow (Y,\sigma)$ is not upper almost contra-I-continuous (resp. upper almost contra-I-continuous) is identical with the union of the I-frontiers of the upper (resp. lower) inverse images of closed sets containing (resp.meeting) F(x).

Proof. Let x be a point of X at which F is not upper almost contra-I-continuous. Then there exists a closed set V of Y containing F(x) such that $U \cap (X \setminus F^+(V)) \neq \emptyset$ for each $U \in IO(X, x)$. Then $x \in I\,Cl(X \setminus F^+(V))$. Since $x \in F^+(V)$, we have $x \in I\,Cl(F^+(Y)$ and hence $x \in IF_r(F^+(A))$. Conversely, let V be any closed set of Y containing F(x) and $x \in IF_r(F^+(V))$. Now, assume that F is upper almost contra-I-continuous at x, then there exists $U \in IO(X, x)$ such that $F(U) \subset V$. Therefore, we obtain $x \in U \subset I\,Int(F^+(V)$. This contradicts that $x \in IF_r(F^+(V))$.

Thus, F is not upper almost contra-I-continuous. The proof of the second case is similar.

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