Sum and Product Theorems on Growth Properties of Bi-complex Entire Functions

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Abstract : Growth properties of entire functions have already been investigated by many authors in the field of advanced complex analysis. Sum and product theorems on order and type of entire functions have been established and extended in many ways. In this paper we wish to establish and extend some of the results over sum and product theorems on order and type of bi-complex entire functions in the field of advanced bi-complex analysis.

Key-words : **Bi-complex number, order, lower order, type, lower type, sum and product of orders, sum and product of types and growth indicators.**

I. INTODUCTION

After the introduction of Bi-complex number by the eminent mathematician C. Segre [9] in 1892, a lot of research work has been done and a huge improvement has been incorporated by the interested mathematicians afterwards. Some of the names of the renowned mathematicians must be mentioned, such as, M. Futagawa [3], E. Hille [5], D. Riley [8], G. B. Price [7] who worked on the improvement of bi-complex algebra and bi-complex analysis for a long time. Growth properties of entire functions related to order, lower order, type, lower type, etc. are widely discussed in the study of advanced complex analysis since last few years. Several results have been established and extended related to sum and product theorems on growth properties of entire functions also. In this paper we have tried to establish and extend some results related to sum and product theorems on the growth properties, such as order and type, relating to bi-complex entire functions in the bi-complex space.

II. DEFINITIONS AND NOTATIONS

We have used some useful definitions and notations as mentioned below in the field of bi-complex entire and meromorphic functions.

Definition 2.1 [9] Bi-complex number

A bi-complex number is defined as $T=\{z_1+i_2z_2 / z_1,z_2\in C(i_1)\}$, where the imaginary units i_1,i_2 follow the rules $i_1^2=i_2^2=-1, i_1i_2=i_2i_1=j$, say and $j^2=1$, *etc.*

Another representation is: $T=\{w_0+w_1i_1+w_2i_2+w_3j / w_i\in R, i=0,1,2,3\}$

Definition 2.2 [9] Idempotent representation of a bi-complex number

Every bi-complex number $(z_1+i_2z_2)$ has the following idempotent representation: $z_1+i_2z_2=(z_1-i_1z_2)e_1+(z_1+i_1z_2)e_2$,

where $e_1=\frac{1+i_1i_2}{2}, e_2=\frac{1-i_1i_2}{2}$.

Definition 2.3 [7] Bi-complex Entire functions

Let U be an open set of T and $w_0\in U$. Then $f:U\subseteq T\to T$ is said to be entire in U if $f'(w_0)\in T$ for all $w_0\in U$, where $\lim_{w\to w_0}\frac{f(w)-f(w_0)}{w-w_0}=f'(w_0)$.

Definition 2.4 [7] Idempotent representation of a bi-complex function

Let X1, X_2 be open sets in $C(i_1)$ and $T\subset C(i_2)$. Then any bi-complex function $f(w)=f(z_1+i_2z_2): X_1\times_e X_2\to T$ can be uniquely represented as follows:

$f(z_1+i_2z_2)=f_{e_1}(z_1-i_1z_2)e_1+f_{e_2}(z_1+i_1z_2)e_2$, for all $z_1+i_2z_2\in X_1\times X_2$, where $f_{e_1}:X_1\to C(i_1)$ and $f_{e_2}:X_2\to C(i_1)$ are two different complex functions.

Definition 2.5 [8] Idempotent representation of a bi-complex entire function

Let X1, X_2 be open sets in $C(i_1)$ and $T\subset C(i_2)$. Then a bi-complex function $f(z_1+i_2z_2)=f_{e_1}(z_1-i_1z_2)e_1+f_{e_2}(z_1+i_1z_2)e_2$ for all $z_1+i_2z_2\in X_1\times_e X_2$, is said to be entire if and only if $f_{e_1}:X_1\to C(i_1)$ and $f_{e_2}:X_2\to C(i_1)$ are entire functions and $f'(z_1+i_2z_2)=f'_{e_1}(z_1-i_1z_2)e_1+f'_{e_2}(z_1+i_1z_2)e_2$.

Definition 2.6 [7] Pole (Strong Pole) of a bi-complex function.

Let f : X → T be a bi-complex meromorphic function on the open set $X\subset T$. We can say that $w=(z_1-i_1z_2)e_1+(z_1+i_1z_2)e_2\in X$ is a (strong) pole for the bi-complex meromorphic function $f(w)=f(z_1+i_2z_2)=f_{e_1}(z_1-i_1z_2)e_1+f_{e_2}(z_1+i_1z_2)e_2$,

if $z_1-i_1z_2\in P_1(X)$ and $z_1+i_1z_2\in P_2(X)$ are poles for $f_{e_1}:P_1(X)\to C(i_1)$ and $f_{e_2}:P_2(X)\to C(i_1)$ respectively.

Proposition 2.1

Let f : X → T be a bi-complex meromorphic function on the open set $X\subset T$. If $w_0\in X$ then w_0 is a pole of f, if and only if $\lim_{w\to w_0}|f(w)|=\infty$.

Definition 2.7 [8] Order of a bi-complex function.

The order ρ (F) of a bi-complex meromorphic function

$F(w)=F_{e_1}(z_1-i_1z_2)e_1+(z_1+i_1z_2)e_2$ is defined as $\rho(F)=Max\{\rho_{F_{e_1}},\rho_{F_{e_2}}\}$

where $\rho_{F_{e_i}}=\limsup_{r_i\to\infty}\dfrac{\log\log M_i(r_i,F_{e_i})}{\log r_i}$ for i = 1,2.

Remark 2.1

The lower order λ(F) of a bi-complex meromorphic function is defined as $\lambda(F)=Min\{\lambda(F_{e_1}),\lambda(F_{e_2})\}$.

where $\lambda_{F_{e_i}}=\liminf_{r_i\to\infty}\dfrac{\log\log M_i(r_i,F_{e_i})}{\log r_i}$ for i = 1,2.

Remark 2.2

The hyper order $\bar{\rho}(F)$ (Hyper lower order $\bar{\lambda}(F)$) and the generalized order $\rho^{(k)}(F)$ (generalized lower order $\lambda^{(k)}(F)$) can also be defined in a similar way.

Definition 2.8 [8] The type of F

The type σ(F) of a bi-complex meromorphic function is defined as $\sigma(F)=Max\{\sigma(F_{e_1}),\sigma(F_{e_2})\}$

where $\sigma(F_{e_i})=\limsup_{r_i\to\infty}\dfrac{\log M_i(r_i,F_{e_i})}{r_i^{\rho_{F_{ei}}}}$ and $0<\rho_{F_{e_i}}<\infty$ for *i=1,2.*

Definition 2.9 [8] The lower type of F

The type σ(F) of a bi-complex meromorphic function is defined as $\sigma(F) = Max\{\sigma(F_{e_1}), \sigma(F_{e_2})\}$

where $\sigma(F_{e_i}) = \liminf_{rI\to\infty} \frac{\log M_i(r_i, F_{e_i})}{r_i^{\rho_{F_{ei}}}}$ and $0 < \rho_{F_{e_i}} < \infty$ for *i=1,2.*

III. LEMMA

In this section we present some lemmas which will be needed in the proof of results.

Lemma 3.1 [11]

If f(z) be an entire function and α *and* β be such that $\alpha > 1$ *and* $0 < \beta < \alpha$, then for all large value of r;

$$M_f(\alpha r) > \beta M_f(r)$$

Lemma 3.2 [10]

If *f* and *g* are any two entire functions then for all sufficiently large values of *r;*

$$M_{fog}(r) > M_f(M_g(r))$$

Lemma 3.3 [2]

Let f(z) and g(z) be any two entire functions of order ρ_f *and* ρ_g respectively. Then

$$(i) \quad \rho_{f+g} = \rho_g \ \ when\ \ \rho_f < \rho_g$$

$$and\ \ (ii) \quad \rho_{f.g} \leq \rho_g \ \ when\ \ \rho_f \leq \rho_g$$, respectively.

Lemma 3.4 [4]

Let f(z) and g(z) be any two entire functions of type σ_f *and* σ_g respectively. Then

$$(i) \quad \sigma_{f+g} \leq \sigma_g \ \ when\ \ \sigma_f < \sigma_g$$

$$and\ \ (ii) \quad \rho_{f.g} \leq \sigma_f + \sigma_g$$, respectively.

IV. THEOREMS

In this section we present our main results of the paper.

Theorem 4.1 {[2], [4]}

If f(w) and g(w) be any two bi-complex entire functions of order ρ_f *and* ρ_g *respectively and if* $\rho_f < \rho_g$ *, then the order* ρ *of F(w) = f(w)+g(w) is equal to* ρ_g.

Proof.

Let us suppose that $\rho_g < \infty$.

Since, for any value of $r < r_0(\varepsilon)$, given $\varepsilon > 0$, we have

$$M_1(r_1; f_{e_1} + g_{e_1}) \leq M_1(r_1; f_{e_1}) + M_1(r_1; g_{e_1})$$

i.e. $$M_1(r_1; f_{e_1} + g_{e_1}) \leq \exp[r_1^{(\rho_{f_{e_1}}+\varepsilon)}] + \exp[r_1^{(\rho_{g_{e_1}}+\varepsilon)}]$$

i.e. $$M_1(r_1; f_{e_1} + g_{e_1}) \leq 2.\exp[r_1^{(\rho_{g_{e_1}}+\varepsilon)}]$$

i.e. $$\frac{\log[\log\{M_1(r_1; f_{e_1} + g_{e_1})\}]}{\log r_1} \leq \rho_{g_{e_1}} + o(1)$$

i.e. $$\lim_{r\to\infty} sup \frac{\log[\log\{M_1(r_1; f_{e_1} + g_{e_1})\}]}{\log r_1} \leq \rho_{g_{e_1}}$$

i.e. $$\rho \ \leq\ \rho_{g_{e_1}} \qquad (4.1)$$

Similarly, in the same way as above, we get $\rho \leq \rho_{g_{e_2}}$ (4.2)

Therefore, from (4.1) and (4.2) and using the definition (2.7), we get

$$\rho \leq \rho_g \tag{4.3}$$

On the other hand, given $\varepsilon > 0$, there is a sequence of numbers $r_n \to \infty$, such that

$$M_1(r_n;\ g_{e_1}) \geq \exp[r_n^{(\rho_{g_{e_1}} - \varepsilon)}]$$

Therefore, $M_1(r_n; f_{e_1} + g_{e_1}) \geq \exp\left[r_n^{(\rho_{g_{e_1}} - \varepsilon)}\right] - \exp[r_n^{(\rho_{f_{e_1}} + \varepsilon)}]$

i.e. $M_1(r_n; f_{e_1} + g_{e_1}) \geq \exp\left[r_n^{(\rho_{g_{e_1}} - \varepsilon)}\right]\left\{1 - \exp[r_n^{(\rho_{f_{e_1}} + \varepsilon)} - r_n^{(\rho_{g_{e_1}} - \varepsilon)}]\right\}$

i.e. $M_1(r_n; f_{e_1} + g_{e_1}) \geq \frac{1}{2}\exp\left[r_n^{(\rho_{g_{e_1}} - \varepsilon)}\right]$, *provided* $(\rho_{f_{e_1}} + \varepsilon) < (\rho_{g_{e_1}} - \varepsilon)$ for that the chosen ε is very small and *n* is sufficiently large.

i.e. $$\frac{\log[\log\{M_1(r_1; f_{e_1} + g_{e_1})\}]}{\log r_1} \geq \rho_{g_{e_1}} + O(1)$$

i.e. $$\limsup_{r\to\infty} \frac{\log[\log\{M_1(r_1; f_{e_1} + g_{e_1})\}]}{\log r_1} \geq \rho_{g_{e_1}}$$

i.e. $$\rho \geq \rho_{g_{e_1}} \tag{4.4}$$

Similarly, in the same way as above, we get $\rho \geq \rho_{g_{e_2}}$ (4.5)

Therefore, from (4.4) and (4.5) and using the definition (2.7), we get

$$\rho \geq \rho_g \tag{4.6}$$

Now, from (4.3) and (4.6), we have $\rho = \rho_g$.

Hence the proof.

Remark 4.1 If $\rho_g \to \infty$, then also $\rho \to \infty$ and the proof is similar.

Remark 4.2 The above theorem does not hold good whenever $\rho_f = \rho_g$.

For an example, let $f(w) = e^w$ *and* $g(w) = -e^w$. Therefore, $\rho_f = \rho_g = 1$ *and* $\rho_{f+g} = 0$.

We have $\rho_f \leq \rho_g$ *implies* $\rho \leq \rho_g$ here.

Corollary 4.1

If f(w) and g(w) be any two bi-complex entire functions of order ρ_f *and* ρ_g *respectively and if* $\rho_f < \rho_g$ *, then the order* $\rho_{f+g} \leq Max\{\rho_f, \rho_g\}$.

Theorem 4.2 [4]

If f(w) and g(w) be any two bi-complex entire functions of order ρ_f *and* ρ_g *respectively and if* $\rho_f \leq \rho_g$ *, then the order* ρ *of F(w) = f(w).g(w) is such that* $\rho \leq \rho_g$.

Proof.

If $\rho_g \to \infty$, then the proof is obvious.

Let us suppose that $\rho_g < \infty$. Now we have,

$$M_1(r_1;\ f_{e_1}.g_{e_1}) \le M_1(r_1; f_{e_1})\ .\ M_1(r_1; g_{e_1})$$

Now, for a given $\varepsilon > 0$ and sufficiently large value of r, we have

i.e. $$M_1(r_1;\ f_{e_1}.g_{e_1}) \le \exp\left[r_1^{(\rho_{f_{e_1}}+\varepsilon)}\right].\ \exp[r_1^{(\rho_{g_{e_1}}+\varepsilon)}]$$

i.e. $$M_1(r_1;\ f_{e_1}.g_{e_1}) \le \exp[2.r_1^{(\rho_{g_{e_1}}+\varepsilon)}]$$

i.e. $$\frac{\log[\log\{M_1(r_1;\ f_{e_1}.g_{e_1})\}]}{\log r_1} \le \rho_{g_{e_1}} + O(1)$$

i.e. $$\lim_{r\to\infty} sup \frac{\log[\log\{M_1(r_1;\ f_{e_1}.g_{e_1})\}]}{\log r_1} \le \rho_{g_{e_1}}$$

i.e. $$\rho \le \rho_{g_{e_1}} \qquad (4.7)$$

Similarly, in the same way as above, we get $$\rho \le \rho_{g_{e_2}} \qquad (4.8)$$

Therefore, from (4.7) and (4.8) and using the definition (2.7), we get

$$\rho \le \rho_g\ .$$

Hence the proof.

Corollary 4.2

If f(w) and g(w) be any two bi-complex entire functions of order ρ_f *and* ρ_g *respectively and if* $\rho_f \le \rho_g$ *, then the order* ρ *of F(w) is such that* $\rho \le Max\{\rho_f\ ,\ \rho_g\}$.

Theorem 4.3 [2]

If f(w) and g(w) be any two bi-complex entire functions of type σ_f *and* σ_g *respectively and if* $\sigma_f < \sigma_g$*, then the type* σ *of F(w) = f(w)+g(w) is such that* $\sigma \le \sigma_g$.

Proof.

Let us suppose that $\sigma_g < \infty$.

Since, for a given $\varepsilon > 0$, there is $R(\varepsilon) > 0$ such that

$$M_1(r_1; f_{e_1} + g_{e_1}) \le M_1(r_1; f_{e_1}) + M_1(r_1; g_{e_1}) \quad for\ \ r > R(\epsilon).$$

i.e. $$M_1(r_1; f_{e_1} + g_{e_1}) \le \exp[\sigma_{f_{e1}} + \epsilon]\ r_1^{\rho_{f_{e_1}}} + \exp\left[\sigma_{g_{e1}} + \epsilon\right] r_1^{\rho_{g_{e_1}}}$$

i.e. $$M_1(r_1; f_{e_1} + g_{e_1}) \le 2.\exp\left[\sigma_{g_{e1}} + \epsilon\right] r_1^{\rho_{g_{e_1}}}$$

i.e. $$\frac{\log\{M_1(r_1; f_{e_1} + g_{e_1})\}}{r_1^{\rho_{f_{e_1}+g_{e_1}}}} \le \sigma_{g_{e_1}} + O(1),\ since\ \ \varepsilon\ is\ arbitrary\ and\ \rho_{f_{e_1}+g_{e_1}} \le \rho_{g_{e_1}}$$

i.e. $$\lim_{r\to\infty} sup \frac{\log\{M_1(r_1; f_{e_1} + g_{e_1})\}}{r_1^{\rho_{f_{e_1}+g_{e_1}}}} \le \sigma_{g_{e_1}}$$

i.e. $$\sigma_{f+g} \le \sigma_{g_{e_1}} \qquad (4.9)$$

Similarly, in the same way as above, we get $$\sigma_{f+g} \le \sigma_{g_{e_2}} \qquad (4.10)$$

Therefore, from (4.9) and (4.10) and using the definition (2.8), we get

$$\sigma_{f+g} \leq \sigma_g \quad i.e., \quad \sigma \leq \sigma_g \tag{4.11}$$

Hence the proof.

Remark 4.3 If $\sigma_g \to \infty$, then also $\sigma_{f+g} \to \infty$ and the proof is similar.

Corollary 4.3

If f(w) and g(w) be any two bi-complex entire functions of type σ_f and σ_g respectively and if $\sigma_f < \sigma_g$, then the type $\sigma_{f+g} \leq Max\{\sigma_f, \sigma_g\}$.

Theorem 4.4 [4]

If f(w) and g(w) be any two bi-complex entire functions of type σ_f and σ_g respectively and if $\sigma_f \leq \sigma_g$, then the type σ of F(w) = f(w).g(w) is such that $\sigma \leq \sigma_f + \sigma_g$.

Proof.

Let us suppose that $\sigma_g < \infty$.

Since, for a given $\varepsilon > 0$, there is $R(\varepsilon) > 0$ such that

$$M_1(r_1; f_{e_1}.g_{e_1}) \leq M_1(r_1; f_{e_1}).\ M_1(r_1; g_{e_1}) \quad for\ r > R(\epsilon).$$

i.e. $$M_1(r_1; f_{e_1}.g_{e_1}) \leq \exp\left[\sigma_{f_{e1}} + \epsilon\right] r_1^{\rho_{f_{e_1}}}.\ \exp\left[\sigma_{g_{e1}} + \epsilon\right] r_1^{\rho_{g_{e_1}}}$$

i.e. $$M_1(r_1; f_{e_1}.g_{e_1}) \leq \exp\left[\sigma_{f_{e1}} + \sigma_{g_{e1}} + 2\epsilon\right].r_1^{\rho_{f_{e_1}}+\rho_{g_{e_1}}}$$

i.e. $$\frac{\log\{M_1(r_1; f_{e_1}.g_{e_1})\}}{r_1^{\rho_{f_{e_1}}+\rho_{g_{e_1}}}} \leq \sigma_{f_{e1}} + \sigma_{g_{e1}} + o(1),\ since\ \varepsilon\ is\ arbitrary.$$

i.e. $$\lim_{r\to\infty} sup \frac{\log\{M_1(r_1; f_{e_1}.g_{e_1})\}}{r_1^{\rho_{f_{e_1}}+\rho_{g_{e_1}}}} \leq \sigma_{f_{e1}} + \sigma_{g_{e1}}$$

i.e. $$\sigma_{f.g} \leq \sigma_{f_{e1}} + \sigma_{g_{e1}} \tag{4.12}$$

Similarly, in the same way as above, we get

$$\sigma_{f.g} \leq \sigma_{f_{e2}} + \sigma_{g_{e2}} \tag{4.13}$$

Therefore, from (4.12) and (24.13 and using the definition (2.8), we get

$$\sigma_{f.g} \leq \sigma_f + \sigma_g \quad i.e., \quad \sigma \leq \sigma_f + \sigma_g \tag{4.14}$$

Hence the proof.

Remark 4.4 If one of σ_f *or* $\sigma_g \to \infty$, then also $\sigma_{f.g} \to \infty$ and the proof is similar.

Corollary 4.4

If f(w) and g(w) be any two bi-complex entire functions of type σ_f and σ_g respectively and if $\sigma_f < \sigma_g$, then the type $\sigma_{f+g} \leq Max\{\sigma_f, \sigma_g\}$.

V. CONCLUSIONS

This field of bi-complex analysis can be developed and extended similarly in the light of advanced Complex analysis in future. The above results can also be extended in case of bi-complex meromorphic functions also. Further results on several growth properties of complex entire and meromorphic functions in the Value distribution theory can be developed and explained in terms of bi-complex numbers also. Interested mathematicians and researchers in this field may be motivated to go through several books and research papers already published around the world.

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REFERENCES

[1] Choi J., Datta S. K., Biswas T. and Sen P. 2015. Sum and product theorems of relative type and relative weak type of entire functions, Honam Mathematical Journal J. 37(1): 65-97.

[2] Clunie J. 1970. The composition of entire and meromorphic functions, Mathematical Essays dedicated to A. J. Macintyre, Ohio University Press: 75-92.

[3] Futagawa M. 1928. On the theory of functions of a quaternary variable, Tôhoku Math. J.29: 175-222.

[4] Hayman W. K. 1964. : Meromorphic Functions, The Clarendon Press, Oxford.

[5] Hille E. 1982. Analytic Function Theory, Chelsea Publishing, New York, Vol. I: 11-308.

[6] Liao L. and Yang C. C. 1999. On the growth of composite entire functions, Yokohama Math. J., Vol. 46: 97-107.

[7] Price, G.B. 1991. An introduction to multi-complex spaces and functions, Marcel Dekker Inc., New York.

[8] Riley J. D. 1953. Contributions to the theory of functions of a bi-complex variable Tôhoku Math. J. 2nd series: 132-165.

[9] Segre C. 1892. Le rappresentazioni reale delle forme complesse'e Gli Enti Iperalgebrici, Math. Ann. Vol. 40: 413-467.

[10] Song G. D. and Yang C. C. 1984. Further growth properties of composition of entire and meromorphic functions, Indian J. Pure Appl. Math, Vol. 15: 67-82.

[11] Valiron G. 1949. Lectures on the general theory of integral functions, Chelsea Publishing Company.