# COMMON FIXED POINT THEOREMS ON BANACH SPACE

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**Abstract:-** In this paper we discussed some fixed point results in Cone Banach Space and introduced the concept of property (E.A.) in Banach Space.

**Keywords:-**Cone Banach space, Common Fixed points, Compatible mapping, weakly compatible, property (E.A.).

## I. INTRODUCTION:-

Huang and Chang [5] gave the notion of cone metric space, replacing the set of real numbers by ordered Banach Space and introduced some fixed point theorems for function satisfying contractive conditions in Banach Spaces. Sh. Rezapour and R. Hamalbarani [10] were generalized result of [5] by omitting the normality condition , which is milestone in developing fixed point theory in cone metric space. After that several articles on fixed point theorems in cone metric space were obtained by different mathematicians such as M. Abbas , G. Junck [7] , D. Ilic [1] etc.

In 2002, Aamri and Moutawakil [6] generalized the notion of noncompatible mapping to the E.A. property. It was pointed out in [6] that the property E.A. buys containment of ranges without any continuity requirements besides minimizing the commutativity conditions of the maps to the commutativity at their points of coincidence. Moreover, the E.A. property allows replacing the completeness requirement of the space with a more natural condition of closeness of the range. Recently, some common fixed point theorems in probabilistic metric spaces fuzzy metric spaces by the E.A. property under weak compatibility have been obtained in [8,3,4].

Some results on fixed point theorems have been extended to Cone Banach Space. E. Karapinar [2] has given some generalizations to this theorems. We will generalize the result of P.G. Vargese [9].

## II. PRELIMINARIES & DEFINITION

**Definition 2.1.** Let $(E, \|\cdot\|)$ be a real Banach space. A subset $P \subseteq E$ is said to be a cone if and only if

(i). P is closed, nonempty and $P \neq \{0\}$

(ii). $a, b \epsilon R, a, b \geq 0,\ x, y \epsilon P$ implies $ax,\ by \in P$

(iii). $P \cap (-P) = \{0\}$

For a given cone $P$ subset of E, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$ while $x << y$ will stand for $y - x \in int\, P$ where $int\, P$ denotes interior of **P** and is assumed to be nonempty.

**Definition 2.2.** Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies

**(i)** $0 \leq d(x, y)$ for every $x, y \epsilon X, d(x, y) = 0$ if and only if $x = y$.

**(ii)** $d(x, y) = d(y, x)$ for every $x, y \epsilon X$.

**(iii)** $d(x, y) \leq d(x, z) + d(z, y)$ for every $x, y, z \epsilon X$.

Then d is a cone metric on $X$ and *(X, d)* is a cone metric space.

**Definition 2.3.** Two self maps A and B on a cone normed space $(X, \|\cdot\|)$ are said to be weak-compatible if they commute at their coincidence points, i.e. $Ax = Bx$ implies $ABx = BAx$.

**Definition 2.4.** Let X be a vector space over R. Suppose the mapping $\|\cdot\| : X \rightarrow E$ satisfies

**(i)** $\|x\| \geq 0$ for all $x \epsilon X$

**(ii)** $\|x\| = 0$ if and only if $x = 0$

**(iii)** $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

**(iv)** $\|kx\| = | k | \|x$ for all $k \in R$.

Then $\|\cdot\|$ is called a norm on $X$, and $(X, \|\cdot\|)$ is called a cone normed space. Clearly each cone normed space is a cone metric space with metric defined by $d(x, y) = \|x - y\|$

**Definition 2.5.** Let $(X, \|\cdot\|)$ be a cone normed space, $x \in X$ and $\{x_n\}$ a sequence in X. Then

**(i).** $\{x_n\}$ converges to x if for every $c \in E$ with $0 \ll c$ there is a natural number N such that .
$$\|x_n - x\| \le c \text{ for all } n \ge N$$

We shall denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

**(ii).** $\{x_n\}$ is a Cauchy sequence, if for every $c \epsilon E$ with $0 \ll c$ there is a natural number N such that
$$\|x_n - x_m\| \le c \text{ for all } n, m \ge N$$

**(iii).** $(X, \|\cdot\|)$ is a complete cone normed space if every Cauchy sequence is convergent. A complete cone normed space is called a Cone Banach space.

**Definition 2.6.** Let F and G be self mappings on a cone normed space $(X, \|\cdot\|)$, they are said to be compatible if $\lim_{n\to\infty} \|FG(x_n) - GF(x_n)\| = 0$ for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} G(x_n) = y$ for some point $y$ in X.

**Proposition 2.7.[10]** Let $(X, \|\cdot\|)$ be a cone normed space. P be a normal cone with constant K. Let $\{x_n\}$ be a sequence in X. Then

**(1)** $\{x_n\}$ converges to $x$ if and only if $\|x_n - x\| \to 0$ as $n \to \infty$.
**(2)** $\{x_n\}$ is a Cauchy sequence if and only $\|x_n - x_m\| \to 0$ as $n, m \to \infty$.
**(3)** if the $\{x_n\}$ converges to $x$ and $\{y_n\}$converges to y then $\|x_n - y_n\| \to \|x - y\|$

**Proposition 2.8.[9]** Let f and g be compatible mappings on a cone normed space $(X, \|\cdot\|)$ such that $\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} G(x_n)$ for some point y in X and for every sequence $\{x_n\}$ in X. Then $\lim_{n\to\infty} F(x_n) = F(y)$. if F is continuous.

## III. Main Result:-

**Theorem 3.1:-** Let F, G, H, and L be mappings on cone Banach Space $(X, ||.||)$ into itself with $\|x\| = d(x, 0)$ satisfying the conditions

$$\|Hx - Ly\| \le a \max\left\{\frac{1}{2}\|F(x) - G(y)\|, \|F(x) - H(x)\|\right\} + b\{\|F(x) - L(y)\| + \|H(x) - G(y)\|\} + c\,\|F(x) - H(x)\| \quad \ldots (1)$$

For all $x, y \in X$ , $a, b, c \ge 0$ $and$ $a + 2b + c < 1$

(a) F and G are onto mapping
(b) F is continuous

Then F, G, H and L have a unique common fixed point.

**Proof:-**

Let $x_0 \in X$ be arbitrary, then $L(x_0) \in X$ , Since F is onto, there is an $x_1 \in X$ , then $L(x_0) \in X$ , since $F$ is onto there is an $x_1 \in X$ such that $(x_1) = L(x_0)$ . Let
$$y_0 = F(x_1) = L(x_0)$$
Again, $x_1 \in X$ since G is onto there is an $x_2 \in X$ such that $G(x_2) = H(x_1)$ . Let
$$y_1 = G(x_2) = H(x_1)$$
Continuing like this we get a sequence $\{y_n\}$ such that
$$y_{2n} = F(x_{2n+1}) = L(x_{2n}) \quad \text{and} \quad y_{2n+1} = F(x_{2n+2}) = H(x_{2n+1})$$

We have
$$\begin{aligned}
\|y_{2n-1} - y_{2n}\| &= \|H(x_{2n-1}) - L(x_{2n})\| \\
&\le a \max\left\{\tfrac{1}{2}\|F(x_{2n-1}) - G(x_{2n})\|, \|F(x_{2n-1}) - H(x_{2n-1})\|\right\} \\
&\quad + b\{\|F(x_{2n-1}) - L(x_{2n})\| + \|H(x_{2n-1}) - G(x_{2n})\|\} \\
&\quad + c\,\|F(x_{2n-1}) - H(x_{2n-1})\| \\
&\le a \max\left\{\tfrac{1}{2}\|y_{2n-2} - y_{2n-1}\|, \|y_{2n-2} - y_{2n-1}\|\right\} \\
&\quad + b\{\|y_{2n-2} - y_{2n}\| + \|y_{2n-1} - y_{2n-1}\|\} + c\|y_{2n-2} - y_{2n-1}\| \\
&\le a\|y_{2n-2} - y_{2n-1}\| + b\{\|y_{2n-2} - y_{2n-1}\| + \|y_{2n-1} - y_{2n}\|\} \\
&\quad + c\|y_{2n-2} - y_{2n-1}\|
\end{aligned}$$
$$(1 - b)\|y_{2n-1} - y_{2n}\| \le (a + b + c)\|y_{2n-2} - y_{2n-1}\|$$

$$\|y_{2n-1} - y_{2n}| \le \frac{(a + b + c)}{(1 - b)}\|y_{2n-2} - y_{2n-1}\|$$
$$\|y_{2n-1} - y_{2n}| \le \lambda\|y_{2n-2} - y_{2n-1}\| \quad \ldots\ldots (2)$$

We have
$$\|y_{2n} - y_{2n+1}\| = \|H(x_{2n}) - L(x_{2n+1})\|$$

$$\leq a \max\left\{\tfrac{1}{2}\left\|F(x_{2n}) - G(x_{2n+1})\right\|, \left\|F(x_{2n}) - H(x_{2n})\right\|\right\}$$
$$+b\left\{\left\|F(x_{2n}) - L(x_{2n+1})\right\| + \left\|H(x_{2n}) - G(x_{2n+1})\right\|\right\}$$
$$+c\,\|F(x_{2n}) - H(x_{2n})\|$$

$$\leq a \max\left\{\tfrac{1}{2}\|y_{2n-1} - y_{2n}\|, \|y_{2n-1} - y_{2n}\|\right\}$$
$$+b\left\{\|y_{2n-1} - y_{2n+1}\| + \|y_{2n} - y_{2n}\|\right\} + c\|y_{2n-1} - y_{2n}\|$$

$$\leq a\|y_{2n-1} - y_{2n}\| + b\left\{\|y_{2n-1} - y_{2n}\| + \|y_{2n} - y_{2n+1}\|\right\}$$
$$+c\|y_{2n-1} - y_{2n}\|$$

$$(1-b)\|y_{2n} - y_{2n+1}\| \leq (a+b+c)\|y_{2n-1} - y_{2n}\|$$

$$\|y_{2n} - y_{2n+1}\| \leq \frac{(a+b+c)}{(1-b)}\|y_{2n-1} - y_{2n}\|$$
$$\|y_{2n} - y_{2n+1}| \leq \lambda\|y_{2n-1} - y_{2n}\| \qquad \ldots\ldots..(3)$$

From (2) and (3)

$$\|y_n - y_{n+1}| \leq \lambda^n\|y_1 - y_0\|$$

For every, Let n>m

$$\|y_n - y_m\| \leq \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \ldots\ldots\ldots.. + \|y_{n-1} - y_n\|$$
$$\|y_n - y_m\| \leq (\lambda^m + \lambda^{m+1} + \cdots\ldots\ldots.. + \lambda^{n-1})\|y_1 - y_0\|$$
$$\|y_n - y_m\| \leq \frac{\lambda^m}{1-\lambda}\|y_1 - y_0\| \qquad \ldots\ldots\ldots\ldots(4)$$

Let $\geq 0$ , then there is a $\delta > 0$ such that $c + N_\delta(0) \subseteq P$ , where $N_\delta(0) = \{y: \|y\| < \delta\}$. Since $\lambda < 1$, for $\delta \geq 0$ there is a positive integer $N$ such that $\frac{\lambda^m}{1-\lambda}\|y_1 - y_0\| \leq \delta$ for $m \geq N$.

Hence $\frac{\lambda^m}{1-\lambda}\|y_1 - y_0\| \in N_\delta(0)$ so $-\frac{\lambda^m}{1-\lambda}\|y_1 - y_0\| \in N_\delta(0)$

Therefore

$$c - \frac{\lambda^m}{1-\lambda}\|y_1 - y_0\| \in c + N_\delta(0) \subseteq P$$

Then $\frac{\lambda^m}{1-\lambda}\|y_1 - y_0\| \leq c$ hence (4) becomes $\|y_n - y_m\| \leq c$ for $n, m \geq N$

Hence $\{y_n\}$ is a Cauchy sequence in $X$ but $X$ is complete therefore there is an $z \in X$ such that $y_n \to z$

$$\lim_{n\to\infty} F(x_{2n+1}) = \lim_{n\to\infty} G(x_{2n}) = \lim_{n\to\infty} H(x_{2n+1}) = \lim_{n\to\infty} L(x_{2n}) = z$$

Since F is continuous

$$\lim_{n\to\infty} F^2(x_n) = F(z)$$

F and H commute, then

$$\lim_{n\to\infty} H\,F(x_{2n+1}) = \lim_{n\to\infty} FH(x_{2n+1}) = F(z)$$

We have

$$\|HF(x_{2n+1}) - L(x_{2n})\| \leq a \max\left\{\tfrac{1}{2}\|F^2(x_{2n+1}) - G(x_{2n})\|, \|F^2(x_{2n+1}) - HF(x_{2n+1})\|\right\} +$$
$$b\left\{\|F^2(x_{2n+1}) - L(x_{2n})\| + \|HF(x_{2n+1}) - G(x_{2n})\|\right\}$$
$$+c\,\|F^2(x_{2n+1}) - HF(x_{2n+1})\|$$

Taking $n \to \infty$ we get

$$\|F(z) - z\| \leq a \max\left\{\tfrac{1}{2}\|F(z) - z\|, \|F(z) - F(z)\|\right\} + b\left\{\|F(z) - z\| + \|F(z) - z\|\right\}$$
$$+\,c\,\|F(z) - z\|$$

$$(1 - a - 2b - c)\|F(z) - z\| \leq 0$$

But $(1 - a - 2b - c) > 0$ So

$$F(z) = z \qquad \ldots\ldots\ldots\ldots.(5)$$

Again

$$\|H(z) - L(x_{2n})\| \leq a \max\left\{\tfrac{1}{2}\|F(z) - G(x_{2n})\|, \|F(z) - H(z)\|\right\}$$
$$+b\left\{\|F(z) - L(x_{2n})\| + \|H(z) - G(x_{2n})\|\right\} + c\,\|F(z) - H(z)\|$$

Taking $n \to \infty$ we get

$$\|H(z) - z\| \leq a \max\left\{\tfrac{1}{2}\|z - z\|, \|z - H(z)\|\right\} + b\left\{\|z - z\| + \|H(z) - z\|\right\} + c\,\|z - H(z)\|$$
$$(1 - a - b - c)\|H(z) - z\| \leq 0$$

But $(1 - a - b - c) > 0$ . So

$$H(z) = z \qquad \ldots\ldots\ldots(6)$$

Since G is onto there is a $u \in X$ such that $z = G(u)$ we have

$$\|HF(x_{2n+1}) - L(x_{2n})\| \le a \max\left\{\frac{1}{2}\|F^2(x_{2n+1}) - G(x_{2n})\|, \|F^2(x_{2n+1}) - HF(x_{2n+1})\|\right\}$$
$$+b\{\|F^2(x_{2n+1}) - L(u)\| + \|HF(x_{2n+1}) - G(u)\|\}$$
$$+c\|F^2(x_{2n+1}) - HF(x_{2n+1})\|$$

Taking $n \to \infty$ we get

$$\|z - L(u)\| \le a \max\left\{\frac{1}{2}\|F(z) - z\|, \|F(z) - F(z)\|\right\} + b\{\|F(z) - L(u)\| + \|F(z) - z\|\}$$
$$+ c\,\|F(z) - H(z)\|$$

From (5) and (6)

$$\|z - L(u)\| \le a \max\left\{\frac{1}{2}\|z - z\|, \|z - z\|\right\} + b\{\|z - L(u)\| + \|z - z\|\} + c\,\|z - z\|$$
$$(1 - b)\|z - L(u)\| \le 0$$

But $(1 - b) > 0$ . So

$$L(u) = z$$

Then $G(u) = L(u) = z$ Since G and L commute then $L(z) = LG(u) = GL(u) = G(z)$.

$$\|H(x_{2n+1}) - L(z)\| \le a \max\left\{\frac{1}{2}\|F(x_{2n+1}) - G(z)\|, \|F(x_{2n+1}) - H(x_{2n+1})\|\right\}$$
$$+b\{\|F(x_{2n+1}) - L(z)\| + \|H(x_{2n+1}) - G(z)\|\}$$
$$+c\,\|F(x_{2n+1}) - H(x_{2n+1})\|$$

Taking $n \to \infty$ we get

$$\|z - L(z)\| \le a \max\left\{\frac{1}{2}\|z - z\|, \|z - z\|\right\} + b\{\|z - L(z)\| + \|z - z\|\}$$
$$+ c\,\|z - z\|$$
$$(1 - b)\|z - L(z)\| \le 0$$

But $(1 - b) > 0$ . So

$$L(z) = z$$
$$L(z) = z = G(z)$$

Hence $F(z) = G(z) = H(z) = L(z) = z$. Clearly $z$ is a fixed point of F, G, H and L

Let $z'$ be a another fixed point of F , G , H and L

$$\|z - z'\| = \|H(z) - L(z')\|$$
$$\le a \max\left\{\frac{1}{2}\|F(z) - G(z')\|, \|F(z) - H(z)\|\right\} + b\{\|F(z) - L(z')\| + \|H(z) - G(z')\|\}$$
$$+ c\,\|F(z) - H(z')\|$$
$$\le a \max\{\frac{1}{2}\|z - z'\|, \|z - z'\|\} + b\{\|z - z'\| + \|z - z'\|\}$$
$$+ c\,\|z - z'\|$$
$$(1 - a - 2b - c)\|z - z'\| \le 0$$
$$z = z'$$

Hence F , G , H and L has a unique common fixed point.

## IV. Fixed point theorem using E.A. property

Aamri and El Moutawakil[6]:-Let F and G be self mapping of a metric space (X,d). We say that f and g satisfy E.A. property if there exist a sequences $\{x_n\}$ in X such that such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

The class of E.A. mappings contains the class of noncompatible mappings.

**Theorem 4.1:-** Let F, G, H, and L be mappings on cone Banach Space $(X, \|.\|)$ into itself with $\|x\| = d(x, 0)$ satisfying the conditions

1. $\|H(x) - L(y)\| \le a \max\left\{\frac{1}{2}\|F(x) - L(y)\|, \|F(x) - H(x)\|\right\}$
   $+b\|H(x) - G(y)\| + c\,\|F(x) - H(x)\|$ …… (7)
   For all $x, y \in X$ , $a, b, c \ge 0$ *and* $a + 2b + c < 1$
2. $H(X) \subseteq G(X)$ and $L(X) \subseteq F(X)$
3. The pairs $(H, F)$ and $(L, G)$ are weakly compatible.
4. One of pair satisfies property E. A.

Then F , G H and L have a unique common fixed point.

**Proof:-** Let $(L, G)$ satisfy E.A. property then by definition there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \mathrm{L}\,\{x_n\} = \lim_{n\to\infty} \mathrm{G}\,\{x_n\} = t \quad \text{for some } t \in X \qquad \text{…… (8)}$$

Since $L(X) \subseteq F(X)$ there exist a sequence $\{y_n\}$ in X such that $L(x_n) = F(y_n)$

Hence $\lim_{n\to\infty} \mathrm{F}\,\{y_n\} = t$ , we claim that $\lim_{n\to\infty} \mathrm{H}\,\{y_n\} = t$ if not , we putting $x = y_n$ and $y = x_n$ in (1)

$$\|H(y_n) - L(x_n)\| \le a \max\left\{\frac{1}{2}\|F(y_n) - L(x_n)\|, \|F(y_n) - H(y_n)\|\right\}$$
$$+b\|H(y_n) - G(x_n)\| + c\,\|F(y_n) - H(y_n)\|$$

From above condition we get

$$||H(y_n) - L(x_n)|| \leq a \max\left\{\frac{1}{2}||L(x_n) - L(x_n)||, ||L(x_n) - H(y_n)||\right\} + b||H(y_n) - G(x_n)|| + c\,||L(x_n) - H(y_n)||$$

$$(1 - a - c)||H(y_n) - L(x_n)|| \leq b||H(y_n) - G(x_n)||$$

Which is contradiction .

Letting $n \to \infty$ we have

$$||H(y_n) - t|| \leq a \max\left\{\frac{1}{2}||t - t||, ||t - H(y_n)||\right\} + b||H(y_n) - t|| + c\,||t - H(y_n)||$$

$$(1 - a - b - c)\,||H(y_n) - t|| \leq 0$$

Hence $\lim_{n\to\infty} \mathrm{H}\,\{y_n\} = \lim_{n\to\infty} \mathrm{F}\,\{y_n\} = t$

Now suppose first $F(X)$ is complete subspace of $X$ then $t = F(u)$ for some $u \in X$ then

$$\lim_{n\to\infty} \mathrm{L}\,\{x_n\} = \lim_{n\to\infty} \mathrm{G}\,\{x_n\} = \lim_{n\to\infty} \mathrm{H}\,\{y_n\} = \lim_{n\to\infty} \mathrm{F}\,\{y_n\} = t = F(u)$$

We claim that $H(u) = F(u)$ if not , we putting $x = u$ and $y = x_n$ in (7)

$$||H(u) - L(x_n)|| \leq a \max\left\{\frac{1}{2}||F(u) - L(x_n)||, ||F(u) - H(y_n)||\right\} + b||H(u) - G(x_n)|| + c\,||F(u) - H(y_n)||$$

From above conditions we get

$$||H(u) - L(x_n)|| \leq a \max\left\{\frac{1}{2}||L(u) - L(x_n)||, ||F(u) - H(u)||\right\} + b||H(u) - G(x_n)|| + c\,||F(u) - H(u)||$$

Letting $n \to \infty$ we have

$$||H(u) - t|| \leq a \max\left\{\frac{1}{2}||t - t||, ||t - H(u)||\right\} + b||H(u) - t|| + c\,||t - H(u)||$$

$$(1 - a - b - c)\,||H(u) - t|| \leq 0$$

$$H(u) = t$$

Clearly $H(u) = F(u) = t$ Hence u is coincidence point of $(H, F)$. Now the weak compatibility $(H, F)$ implies that $HF(u) = FH(u)$ or $Ht = Ft$

On the other hand $H(X) \subseteq G(X)$ there exist $v \in X$ such that $H(u) = G(v)$. Thus

$$H(u) = F(u) = G(v) = t$$

Let us show that $v$ is coincidence point of (L,G) that is $G(v) = L(v) = t$ if not then putting $x = u$ and $y = v$ in (1) we have

$$||H(u) - L(v)|| \leq a \max\left\{\frac{1}{2}||F(u) - L(v)||, ||F(u) - H(u)||\right\} + b||H(u) - G(v)|| + c\,||F(u) - H(u)||$$

From above conditions we get

$$||H(u) - L(v)|| \leq a \max\left\{\frac{1}{2}||H(u) - L(v)||, ||H(u) - H(u)||\right\} + b||H(u) - G(v)|| + c\,||H(u) - H(u)||$$

Letting $n \to \infty$ we have

$$||t - L(v)|| \leq \frac{a}{2}||t - L(v)|| + b||t - t|| + c\,||t - t||$$

$$(1 - \frac{a}{2})\,||t - L(v)|| \leq 0$$

$$L(v) = t$$

Clearly $L(v) = G(v) = t$ Hence$v$ is coincidence point of $(L, G)$. Now the weak compatibility of pair $(L, G)$ implies that $GL(u) = LG(u)$ or $Lt = Gt$. Therefore t is common coincidence of $F, G, H$ and $L$.

In order to show that t is a common fixed point, let us t is a common fixed point of $F, G, H$ and $L$, Let us put $x = u$ and $y = t$ in (1) we have

$$||H(u) - L(v)|| \leq a \max\left\{\frac{1}{2}||F(u) - L(t)||, ||F(u) - H(u)||\right\} + b||H(u) - G(t)|| + c\,||F(u) - H(u)||$$

From above conditions we get

$$||t - L(t)|| \leq a \max\left\{\frac{1}{2}||F(u) - L(t)||, ||F(u) - H(u)||\right\} + b||H(u) - G(t)|| + c\,||F(u) - H(u)||$$

$$||t - L(t)|| \leq \frac{a}{2}||t - L(t)|| + b||t - L(t)|| + c\,||t - t||$$

$$\left(1 - \frac{a}{2} - b\right)||t - L(t)|| \leq 0$$

$$L(t) = t$$

Clearly $F(t) = H(t) = L(t) = G(t) = t$

Hence $t$ is common fixed of $F, G, H$ and $L$.

Similar argument arises if we assume that $G(X)$ is complete subspace of X. Similarly the property (E.A.) of the pair (H,F) will give similar result.

For uniqueness of common fixed point, let us assume that w be another common fixed point of F,G,H and L. Let us put $x = w$ and $y = t$ in (1) we have

$$||H(w) - L(t)|| \leq a \max\left\{\frac{1}{2}||F(w) - L(t)||, ||F(w) - H(w)||\right\} + b||H(w) - G(t)|| + c\,||F(w) - H(w)||$$

From above conditions we get

$$||w - t|| \leq a \max\left\{\frac{1}{2}||w - t||, ||w - w||\right\} + b||w - t|| + c\,||w - w||$$

$$||w - t)|| \leq \frac{a}{2}||w - t|| + b||w - t||$$

$$\left(1 - \frac{a}{2} - b\right)||w - t|| \leq 0$$

$$w = t$$

Clearly $F(t) = H(t) = L(t) = G(t) = t$

Hence $t$ is common fixed of $F, G, H$ and $L$. This completes the proof.

**Corollary 4.2:-** Let F and H be mappings on cone Banach Space $(X, ||.||)$ into itself with $||x|| = d(x, 0)$ satisfying the conditions

$$||H(x) - H(y)|| \leq a \max\left\{\frac{1}{2}||F(x) - H(y)||, ||F(x) - H(x)||\right\} + b||H(x) - F(y)|| + c\,||F(x) - H(x)|| \quad \dots\dots(9)$$

For all $x, y \in X$, $a, b, c \geq 0$ and $a + 2b + c < 1$

1. $H(X) \subseteq F(X)$
2. The pair $(H, F)$ weakly compatible.

Then F and H have a unique common fixed point.

**Proof:-** Let $(H, F)$ satisfy E.A. property then by definition there exist a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} \mathrm{H}\{x_n\} = \lim_{n\to\infty} \mathrm{F}\{x_n\} = t$ for some $t \in X$ ……(10)

Now suppose first $F(X)$ is complete subspace of $X$ then $\lim_{n\to\infty} \mathrm{F}\{x_n\} = F(u)$ for some $u \in X$ then

$$\lim_{n\to\infty} \mathrm{H}\{x_n\} = \lim_{n\to\infty} \mathrm{F}\{x_n\} = F(u)$$

We claim that $H(u) = F(u)$ if not , we putting $x = x_n$ and $y = u$ in (1)

$$||H(x_n) - H(u)|| \leq a \max\left\{\frac{1}{2}||F(x_n) - H(u)||, ||F(x_n) - H(x_n)||\right\} + b||H(x_n) - F(u)|| + c\,||F(x_n) - H(x_n)||$$

Letting $n \to \infty$ yields

$$||F(u) - H(u)|| \leq a \max\left\{\frac{1}{2}||F(u) - H(u)||, ||F(u) - F(u)||\right\} + b||F(u) - F(u)|| + c\,||F(u) - F(u)||$$

$$||F(u) - H(u)|| \leq \frac{a}{2}||F(u) - H(u)||$$

$$\left(1 - \frac{a}{2}\right)||F(u) - H(u)|| \leq 0$$

Clearly $H(u) = F(u)$

From (2) $H(u) = F(u) = t$. Hence $t$ is coincidence point of $(F, H)$ .Now from the weak compatibility of pair $(H, F)$ implies that $HF(u) = FH(u)$ or $Ht = Ft$. We have to show that $t$ is common fixed point of $F$ and $H$. Let us put $x = u$ and $y = t$ in (1)

$$||H(u) - H(t)|| \leq a \max\left\{\frac{1}{2}||F(u) - H(t)||, ||F(u) - H(u)||\right\} + b||H(u) - F(t)|| + c\,||F(u) - H(u)||$$

$$||H(u) - H(t)|| \leq a \max\left\{\frac{1}{2}||F(u) - H(t)||, ||F(u) - H(u)||\right\} + b||H(u) - F(t)|| + c\,||F(u) - H(u)||$$

Letting $n \to \infty$

$$||t - H(t)|| \leq \frac{a}{2}||t - H(t)|| + b||t - H(t)||$$

$$\left(1 - \frac{a}{2} - b\right) \left|\left|t - H(t)\right|\right| \leq 0$$
$$H(t) = t$$

Clearly $F(t) = H(t) = t$

Hence $t$ is common fixed of $F$ and $H$. Uniqueness of the common fixed point follows easily.

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