SELF-ADJOINT BOUNDARY VALUE PROBLEMS ASSOCIATED TO SCHRODINGER OPERATORS ON FINITE NETWORK

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Abstract:
We aim here at analyzing self-adjoint boundary value problems on finite networks associated with positive semi-definite Schrodinger operators. Also, we study the existence and uniqueness of solutions and its variation formulation. And, we aim at analyzing different types of boundary value problem associated with the Schrodinger operator with ground state q.

***IndexTerms* - The First Green Identity, The Second Green Identity, Self-adjoint boundary value problems, Euler identity.**

I. INTRODUCTION

In this paper, we analyze self-adjoint boundary value problems on finite networks associated with positive semi-definite Schrodinger operators. Among others, we treat general mixed boundary value problems that include the well-known Dirichlet and Neumann problems and also the Poisson equation.

The first of those papers are concerned with the general analysis of self-adjoint boundary value problems associated with non-negative variations of the combinatorial Laplacian and its associated Green functions from a point of Potential Theory. A Schrodinger operator on a finite network is a linear operator of the form $L_q = L$ + q, where L is the combinatorial Laplacian of the network and q is a function on the vertex set.

So, a Schrodinger operator may be seen as a variation of the combinatorial Laplacian. Some of the authors obtained a generalization of this result when the ground state takes negative values, which was applied to the study of Dirichlet problems and Poisson equations. Here we extend the above results to the energy associated with general self-adjoint Boundary value problem. In particular, we show that any Boundary value problem has a unique solution provided that its associated energy is positive definite and we characterize when this happens in terms of the ground state.

Preliminaries

Along with the paper, Γ = (V, E) denotes a simple, finite and connected graph without loops, with vertex set V and edge set E. Two different vertices, x, y ∈ V, are called **adjacent**, which will be represented by x ∼ y, if {x, y} ∈ E.

Given x, y ∈V, if d(x, y) is the length of the shortest path joining x and y it is well-known that d defines a distance on the graph. Given a vertex subset G⊂ V, we denote by $\boldsymbol{G^c}$ its complementary in V and we call boundary and closure of G, the sets δ (G) = {x ∈ V: d(x, G) = 1} and $\bar{G}$= G ∪ δ (G), respectively. Clearly, $\bar{G}$={x ∈ V: d(x, G) ≤ 1}.

Support of u:
The sets of functions and non-negative functions on V are denoted by C(V) and C^{+}(V) respectively. If u ∈ C (V), its support is given by sup(u) = {x ∈ V : u(x) ≠ 0}.

Moreover, if G is a non-empty subset of V, its characteristic function is denoted by χ_G and we can consider the sets C(G)= {u ∈ C(V) : supp(u) ⊂ G} and C^{+}(G) = C(G) ∩ C^{+}(V). For any u ∈C(G),

We denote by $\int_G u(x)dx$ the value $\sum_{x\in G} u(x)$. We call weight on G any function σ ∈ C^{+}(G) such that supp(σ) = G. The set of weights on G is denoted by C^{*}(G)

Conductance:

The conductance on Γ is a function c: V × V ⟶ $\mathbb{R}^{+}$ such that c(x, y) > 0 if and only if x ∼ y. We call network any pair (Γ, c), where c is a conductance onΓ. The network (Γ, c) is simply referred by Γ.

Combinatorial Laplacian:

The combinatorial Laplacian or only the Laplacian of the system Γ is the linear operator L:C(V)⟶C(V) that assigns to each u ∈ C(V) the function

(A) $$L(x) = \int_V c(x,y)\,((u(x) - u(y))\,dy,\ x \in V.$$

Normal Derivative:

If G is a proper subset of V, for each u ∈ C($\bar{G}$), we define the normal derivative of u as the function in C(δ(G)) given by

(B) $$\left(\frac{\partial u}{\partial n_G}\right)(x) = \int_G c(x,y)(u(x) - u(y))\,dy,\ \text{for any } x \in \delta(G).$$

The relation between the values of the Laplacian on G and the values of the normal derivative at δ(G) is given by **the First Green Identity**

$$\int_G v\,L(u)\,dx = \frac{1}{2}\int_{\bar{G}}\int_{\bar{G}} c\,(x, y)\,(u(x) - u(y))(v(x) - v(y))\,dxdy - \int_{\delta(G)} v\frac{\partial u}{\partial n_G}\,dx,$$

Where u, v ∈ C ($\bar{G}$) and c_G= c. $\chi_{(G\times G)\setminus(\delta(G)\times\delta(G))}$.

4A direct consequence of the above identity is the so-called **The Second Green Identity**

$$\int_G (vL(u) - uL(v))dx = \int_{\delta(G)} \left(u\frac{\partial v}{\partial n_G} - v\frac{\partial u}{\partial n_G}\right)dx,\ \text{for all } u, v \in C(\bar{G}).$$

When G = V the above identity tells us that the combinatorial Laplacian is a self-adjoint operator and that $\int_V L(u)\,dx = 0$ for any u ∈ C(V). Moreover, since Γ is connected L(u) = 0 if and only if u is a constant function.

Schrodinger Operator:

Given q ∈ C (V) the Schrodinger operator on Γ with ground state q is the linear operator L_q: C (V) → C (V) that assigns to each u ∈ C (V) the function L_q (u) = L(u)+qu.

SELF ADJOINT BOUNDARY VALUE PROBLEM

Here we study a different type of boundary value problems associated with the Schrodinger Operator with ground state q. Given a non-empty subset G ⊂ V , δ(G) = $H_1 \cup H_2$, where $H_1 \cap H_2 = \emptyset$ and functions g ∈ C(G), g_2 ∈ C(H_2), g_1 ∈ C(H_1), a boundary value problem on G consists on finding u ∈ C($\bar{G}$) such that

(*) $$L_q(u) = g \text{ on } G,\quad \frac{\partial u}{\partial n_G} + qu = g_1 \text{ on } H_1,\quad \text{and } u = g_2 \text{ on } H_2,$$

In addition, the associated homogeneous boundary value problem consists on finding u ∈ C($\bar{G}$) such that L_q (u) = 0 on F, $\frac{\partial u}{\partial n_G}$+ qu=0 on H_1 and u = 0 on H_2.

The Green Identity implies that the boundary value problem (*) is self-adjoint in the sense that $\int_G v\,L_q(u)dx = \int_G u\,L_q(v)dx$ dx for all u, v ∈ C(G ∪ H_1) verifying that $\frac{\partial u}{\partial n_G}$+ qu = $\frac{\partial v}{\partial n_G}$ + qv = 0 on H_1.

Problem (*) is generally called as a mixed Dirichlet-Robin problem and summarizes the different boundary value problems that appear in the literature with the following proper names:

(i) Dirichlet problem: $\emptyset \neq H_2 \neq \delta(G)$ *and hence* $H_1 = \emptyset$.

(ii) Robin problem: $\emptyset \neq H_1 = \delta(G)$ *and* $q \neq 0$ *on* H_1.

(iii) Neumann problem: $\emptyset \neq H_1 = \delta(G)$ *and* $q = 0$ *on* H_1.

(iv) Mixed Dirichlet-Neumann problem: $H_1, H_2 \neq \emptyset$ *and* $q = 0$ *on* H_1.

(v) Poisson equation on V: G = V.

In this paper, we extend the above results for the self-adjoint boundary value problem (*).

Proposition 1.1 *(Fredholm Alternative)* given $g \in C(G)$, $g1 \in C(H_1)$, $g2 \in C(H_2)$, *the boundary value problem* $L_q(u) = g$ *on* G, $\frac{\partial u}{\partial n_G} + qu = g_1$ *on* H_1 & $u = g_2$ *on* H_2 *has a solution if and only if for any* $v \in C(\bar{G})$*The solution of the homogeneous problem it is verified* $\int_G gv dx + \int_{H_1} g_1 v dx = \int_{H_1} g_2 \frac{\partial v}{\partial n_G} dx$.

In addition, when the above condition holds, then there exists a unique $u \in C(\bar{G})$ *The solution of the boundary value problem such that* $\int_{\bar{G}} uv\, dx = 0$*, for any* $v \in C(\bar{G})$ *The solution of the homogeneous problem.*

Proof

Observe that problem (*) is equivalent to the boundary value problem $L_q(u) = g - L_q(g_2)$ on G, $\frac{\partial u}{\partial n_G} + qu = g_1$ on H_1 and u = 0 on H_2 in the sense that u is a solution of this problem if and only if u + g_2 is a solution of (*).

Consider now the linear operator $\mathcal{F}$: C(G ∪ H_1) → C(G ∪ H_1) defined as $\mathcal{F}(u) = L_q(u)$ on G and $\mathcal{F}(u) = \frac{\partial u}{\partial n_G} + qu$ on H_1. If γ denotes the space of solutions of the homogeneous problem, then ker $\mathcal{F} = \gamma$.

Moreover, from the Second Green Identity, we get that $\int_{G \cup H_1} v\mathcal{F}(u)\, dx = \int_{G \cup H_1} u\mathcal{F}(v)\, dx$;

i.e., $\mathcal{F}$ is self-adjoint and hence Img $\mathcal{F} = \gamma^{\perp}$, using the classical Fredholm Alternative.

Consequently problem (*) has a solution if and only if the function $\hat{g} \in C(G \cup H_1)$ given by $\hat{g} = g - L_q(g_2)$ on G and $\hat{g} = g_1$ on H_1verifies that

$$0 = \int_{G \cup H_1} \hat{g} v\, dx = \int_G gv dx + \int_{H_1} g_1 v dx\ dx - \int_G v\, L_q(g_2) dx$$
$$= \int_G gv dx + \int_{H_1} g_1 v dx\ dx - \int_{H_2} g_2 \frac{\partial v}{\partial n_G} dx, \text{ for any } v \in \gamma.$$

Finally, the Fredholm Alternative also establishes that when the necessary and sufficient condition is attained there exists a unique w ∈ $\gamma^{\perp}$, such that $\mathcal{F}(w) = \hat{g}$. Therefore, u = w + g_2 is the unique solution of problem (*) such that for any v ∈ γ

$$\int_{\bar{G}} uv\, dv = \int_{G \cup H_1} uv\, dv = \int_{G \cup H_1} wv\, dv = 0,$$

since v = 0 on H_2 and g_2 = 0 on G ∪ H_1.

Fredholm Alternative establishes that the existence of solution of problem (*) for any data g, g_1 andg_2 is equivalent to the uniqueness of the solution and hence it is equivalent to the fact that the homogeneous problem has v = 0 as its unique solution.

So, applying the First Green Identity, if v ∈ γ

$$0 = \int_G v\, L_q(v)\, dx = \frac{1}{2} \int_{\bar{G}} \int_{\bar{G}} c_G(x, y)(v(x) - v(y))^2\, dxdy + \int_{\bar{G}} qv^2 dx$$

and hence uniqueness is equivalent to be v = 0 the unique solution of the above equality.

The above equality leads to define the **energy** associated with the Problem (*) as the symmetric auxiliary form ξ_q^G : C($\bar{G}$) × C($\bar{G}$) → ℝ given for any u, v ∈ C($\bar{G}$) by

(**) $$\xi_q^G(u,v) = \frac{1}{2} \int_{\bar{G}} \int_{\bar{G}} c_G(x, y)\,(u(x) - u(y))((v(x) - v(y))\, dxdy + \int_{\bar{G}} quvdx.$$

A sufficient condition so that the homogeneous problem associated with (*) have v = 0 as its unique solution is that the energy is positive definite. Next, we characterize when this property is achieved.

To do this, it will be useful to introduce for any weight σ on $\bar{G}$, the so-called ground state associated with σ as $q_\sigma = -\frac{1}{\sigma} L(\sigma)$ on G, $q_\sigma = -\frac{1}{\sigma} \frac{\partial \sigma}{\partial n_G} \delta(G)$ and q_σ = q otherwise.

Clearly, if σ ∈ $C^*(\bar{G})$ then for any a > 0, μ = aσ ∈ $C^*(\bar{G})$ and moreover $q_\mu = q_\sigma$. Observe that q_σ= 0 iff σ = a$\chi_{\bar{G}}$, with a > 0.

More generally, if σ ∈ C^*(G), then taking v =$\chi_{\bar{G}}$ in the Second Green, an identity we obtain that $\int_{\bar{G}} \sigma q_\sigma = 0$, which implies that q_σ must take positive and negative values, except when σ = a$\chi_{\bar{G}}$, a > 0. Moreover,

it was proved that $-\int_{\bar{G}} c_G(x, y)\, dy < \sigma q_\sigma(x)$ for any x ∈$\bar{G}$ and also that when $H_2 \neq \emptyset$, then it is possible to choose σ ∈ C^*(G), such that $q_\sigma(x) < 0$ for any x ∈ F ∪ H_1.

Proposition1.2. *The Energy* ξ_q^G *is positive semi-definite if and only if there exists* $\sigma \in C^*(G)$*, such that* $q \geq q_\sigma$. *Moreover, it is not strictly definite if and only if* $q = q_\sigma$*, in which case* $\xi_q^G(v, v) = 0$ *if and only if* $v = a\,\sigma$, $a \in \mathbb{R}$.

Proof.

Consider the network $\Gamma_G = (\bar{G}, \bar{E}, C_G)$, where E = {(x, y) ∈ E : C_G (x, y) > 0} and let L its combinatorial Laplacian. Then, for any u ∈ C($\bar{G}$), L(u) = $\bar{L}$(u) on G and L(u) = $\frac{\partial u}{\partial n_G}$ on δ(G).Moreover, ξ_q^G(u, u) = $\int_{\bar{G}}$ u$\bar{L}$(u) dx + $\int_{\bar{G}}$ qu^2dx and hence the results follow by applying Proposition 1.3.

The next result establishes the fundamental result about the existence and uniqueness of solution for Problem (*) and about its variation formulation.

Proposition 1.3 *(Dirichlet principle) Suppose that there exists σ ∈ C^*(G) such that q ≥ q_σ. Given g ∈ C(G), g_1 ∈ C(H_1), g_2 ∈ C(H_2), consider the convex set C_{g_2} = {v ∈ C(F) : v = g_2 on H_2} and the quadratic functional $\mathcal{J}_q$: C($\bar{G}$) → ℝ determined by the expression $\mathcal{J}_q$ (u)= $\frac{1}{2}\int_{\bar{G}} \int_{\bar{G}} c_G(x, y)(u(x) - u(y))^2\, dxdy + \int_{\bar{G}} qu^2dx$ -2 $\int_G gu\, dx$ -$\int_{H_1} g_1udx$. Then u ∈ C($\bar{G}$) is a solution of (*) if and only if u minimizes $\mathcal{J}_q$ on C_{g_2}. Moreover, if it is not simultaneously true that $H_2 = \emptyset$ and q = q_σ, then $\mathcal{J}_q$ has a unique minimum on C_{g_2}. Otherwise, $\mathcal{J}_q$ has a minimum if and only if $\int_G g\sigma dx + \int_{\delta(G)} g_1\sigma dx = 0$. In this case, there exists a unique minimum u ∈ C($\bar{G}$) such that $\int_{\bar{G}} u\sigma dx = 0$.*

Proof.

Observe that C_{g_2} = g_2 + C(G∪H_1) and that for all v ∈ C(G∪H_1), we get,$\mathcal{J}_q$ (v)= ξ_q^G(v, v) - 2 $\int_G$ gv dx -$\int_{H_1}$ $g_1$vdx.
Keeping in mind, that q ≥ q_σ, we get that $\mathcal{J}_q$ is a convex functional on C(G∪H_1) and hence on C_{g_2}.
Moreover, it is an strictly convex functional if and only if it is not simultaneously true that $H_2 = \emptyset$ and q = q_σ and then $\mathcal{J}_q$ has a unique minimum on C_{g_2}.
On the other hand, when $H_2 = \emptyset$ and q = q_σ, simultaneously the minima of $\mathcal{J}_q$ are characterized by the **Euler identity**:

ξ_q^G(u, v) = $\int_G$ gv dx +$\int_{H_1}$ g_1v dx, for all v ∈ C($\bar{G}$) Since in this case ξ_q^G(u, σ) = 0, for all u ∈ C($\bar{G}$) necessarily g and g_1 must satisfy that $\int_G$ gσ dx +$\int_{H_1}$ $g_1\sigma$ dx = 0.

Moreover, if this condition holds and V denotes the vector subspace generated by σ, then u ∈ $V^\perp$ minimizes $\mathcal{J}_q$ on $V^\perp$ if and only if u minimizes $\mathcal{J}_q$ on C($\bar{G}$) and the existence of minimum follows since $\mathcal{J}_q$ is strictly convex on $V^\perp$.

In any case, the equations described in (*) are the Euler-Lagrange identities for the corresponding minimization problem. The following result is an extension of the monotonicity property of the Schrodinger operator in the case q ≥ q_σ.

Proposition1.4. *Suppose that q ≥ q_σ, and that it is not simultaneously true that $H_2 = \emptyset$ and q = q_σ. If u ∈ C($\bar{G}$) verifies that L_q (u) ≥ 0 on G, $\frac{\partial u}{\partial n_G}$+ qu ≥ 0 on H_1 and u ≥ 0 on H_2, then u ∈ $C^+(\bar{G})$*

Proof.

Consider again the network Γ_G = ($\bar{G}$,$\bar{E}$,C_G), where $\bar{E}$ = {(x, y) ∈E: C_G (x, y) > 0} and let $\bar{L}$ its combinatorial Laplacian. Then, if u ∈ C($\bar{G}$) verifies the hypotheses, $\bar{L}$ (u) ≥ 0 on G∪H_1 and the conclusion follows by applying Proposition 4.1 in [3].

Suppose that there exists σ ∈ C^*(G), such that q ≥ q_σ, and it is not simultaneously true that $H_2 = \emptyset$ and q = q_σ. The Green operator associated with Problem (*) is the linear operator $\mathcal{G}_q^F$: C(G) → C($\bar{G}$)that assigns to any g ∈ C(G) the unique solution of the boundary value problem L_q(u) = g on G, $\frac{\partial u}{\partial n_G}$+ qu = 0 on H_1 and u = 0 on H_2.

Moreover, we define the Green function associated to Problem (*) as the function G_q^F : $\bar{G}$×G → ℝ that assigns to any y ∈ G and any x ∈ $\bar{G}$ the value G_q^F (x, y) = $\mathcal{G}_q^F$ (ε_y)(x), where ε_y stands for the Dirac function at y.

So, for any g ∈ C(G) it is verified that $\mathcal{G}_q^F$(g)(x) = G_q^F(x, y) g(y) dy. Finally, let us remark that from the above proposition $G_q^F \geq 0$ and moreover G_q^F(x, y) = G_q^F(y, x) for any x, y ∈ G, since the boundary value problem (*) is self-ad joint.

Conclusion:

In this paper, we analyzed the self-adjoint boundary value problems on finite networks associated with positive semi-definite Schrodinger operators. Among others, we treated general mixed boundary value problems that include the well-known Dirichlet and Neumann problems and also the Poisson equation. Some of the authors obtained a generalization of this result when the ground state takes negative values, which was applied to the study of Dirichlet problems and Poisson equations. Here we extended the above results to the energy associated with general self-adjoint Boundary value problem. In particular, we showed that any Boundary value problem has a unique solution provided that its associated energy is positive definite and we characterize when this happens in terms of the ground state.

REFERENCES

1. E. Bendito, A. Carmona, A. M. Encinas: Solving boundary value problems on networks using equilibrium measures. J. Funct. Anal., 171 (2000), 155–176.
2. E. Bendito, A. Carmona, A. M. Encinas: Equilibrium measure, Poisson kernel and effective resistance on networks in “Random Walks and Geometry”, (V. Kaimanovich, K. Schmidt and W. Woess ed.), 363–376, Walter de Gruyter, 2004.
3. E. Bendito, A. Carmona, A. M. Encinas: Potential Theory for Schrodinger operators on finite networks. Rev. Mat. Iberoamericana, 21 (2005), 771–818.
4. A. Bensoussan, J. L. Menaldi: Difference equations on weighted graphs. J. Convex Anal., 12 (2005), 13–44.
5. S.-Y. Chung, C. A. Berenstein: ω-harmonic functions and inverse conductivity problems on networks. SIAM J. Appl. Math., 65 (2005), 1200–1226.
6. Anandam, Harmonic Functions, and Potentials on Finite or Infinite Networks, Lecture Notes of the Unione Mathematica Italiana 12, Springer-Verlag, 2011.