ON THE EXISTENCE OF AFFINE MOTION IN TACHIBANA RECURRENT SPACES

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ABSTRACT

In the present paper, we study the affine motions in Tachibana recurrent spaces by taking an infinitesimal transformation and derive some important theorems.

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1.Introduction. A$_n$n(=2m) dimensional Tachibana space T_n^c is a Riemannian space, which admits a tensor field F_i^h satisfying

$$F_j^h F_h^i = -\delta^i_j, \qquad \ldots(1.1)$$

$$F_{i,j} = -F_{ji}, (F_{ji} = F^{\alpha}_i g_{aj}) \qquad \ldots(1.2)$$

And

$$F^h_{j,k} = 0, \qquad \ldots(1.3)$$

Where the comma (,) followed by an index denotes the operation of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemanian curvature tensor field R^i_{jkl} is defined by

$$R^i_{jkl} = \partial\Gamma^i_{kl} = \partial\Gamma^i_{jl} + \Gamma^i_{ja}\Gamma^a_{kp} = \Gamma^i_{ka} r^a_{ji} \qquad \ldots(1.4)$$

If the space T^c_n satisfies conditions

$$R^i_{jkl,a} = \lambda_a R^i_{jkl}, \lambda_a \neq 0. \qquad \ldots(1.5)$$

It will be called a Tachibana recurrent space and will be denoted by * T^c_n..For any tensor $B^{i..}_{jk...}$ in the space T^c_n or * T^c_n., we can find the formula

$$£_v(B^{i..}_{jk..b}) - (£_v B^{j..}_{jk..}),b = B^{a.}_{jk..}(£_v \Gamma^i_{ab}) + \ldots - B^{i.}_{ak.}((£_v \Gamma^i_{jb}) - B^{i..}_{ja..}(£_v \Gamma^i_{kb}) \ldots, \qquad \ldots(1.6)$$

Where $£_v$ denotes the Lie-derivative with respect to the infinitesimal transformation

$x^{-1} = x^i + v^i(x)\delta t$,

where δt is an infinitesimal constant. The above infinitesimal transformation, considered at each point of T^c_n, is called an affine motion, when and only when

$£_v \Gamma^i_{jk} = 0.$

According to Knebleman ([1], 1929, [2], 1945) and Slebedzinski ([3], 1932), for an affine motion, the two operators $£_v$ covariant operator (,) are commutative with each other.

Making use of $£_v \Gamma^i_{jk} \neq 0$, we have

$£_v R^i_{jkl} = 0.$ …(1.7)

Applying $£_v$ on the both sides of (1.5) and using (1.6) and (1.7), we get

$(£_v\lambda n) R^i_{jkl} = 0,$ …(1.8)

i.e., the Recurrence vector λa of the space must be a Lie-invariant one. The space * T^c_n, admitting an infinitesimal transformation $x^{-1}= x^i + v^i (x)\delta t$, which satisfies (1.8) will be called a restricted space, or briefly an S- * T^c_n space.

We, now prove the following **Lemma**. In an S-* T^c_n space, if the recurrence vector λn is gradient ne, then

$\lambda n v^a$ = constant.

Proof. Let us put $\alpha = \lambda a v^a$, then, from the basic condition

$£_v\lambda a = v^a\lambda a,b + \lambda b v^b{}_{.a},$

And the assumption $\lambda a,b = \lambda b,a$, we see that $\alpha_b = 0$.

This completes the proof

In an S- * T^c_n space, in view of (1.5) and the definition of Lie-derivative, we get

$£_v R^i_{jkl} = \alpha R^i_{jkl} – R^r_{jkl}v^i_{,r} + R^r_{jkl}v^r_{,k}+ R^i_{jkr}v^r_{,l}$ …(1.9)

Calculating $(R^i_{jkl,ba} – R^i_{jkl,ab})$, we have the following Ricci-identity:

$R^i_{jkl,ba} – R^i_{jkl,ab} – R^r_{jkl} R^i_{rab} + R^i_{rkl} R^r_{jab} + R^i_{jrl} R^r_{kab} + R_{jkr} R^r_{lab} = 0$ …(1.10)

Next, let us assume that α is not a constant, then from then from the above Lemma, we see that

$\lambda ab = \lambda a,b – \lambda b,a \neq 0.$

Let us take $v^i_{,j} = R^i_{jkl} f^{kl}$ for a suitable non-symmetric tensor f^{kl}

Multiplying (1.10) by f^{ab} side by side and summing over a and b, we have

$F^{ab} A_{ab} R^i_{jkl} = R^r_{jkl} v^i_{,r} – R^i_{rkl} v^r_{,j} – R^i_{jrl} v^r_{,k} – R^i_{jkr} v^r_{,1.}$ …(1.11)

Comparing equations (1.9) and (1.11), we get

$£_v R^i_{jkl} = (\alpha – A_{ab} f^{ab}) R^i_{jkl},$

Which vanishes, if and only if, the curvature tensor has the following resolved from:

$\alpha R^i_{jkl} = A_{kl} v^i_{,j}$. …(1.12)

we have the following

Definition (1.1). An S-*T^c_n space satisfying $cV^a \neq$ constant is called a special Tachibaba space of the second kind.

Definition (1.2). An S-*T^c_n space satisfying $\lambda a V^a$ = constant, is called a special Tachibana space of the second kind.

In order that we have (1.12), the condition

$R^i_{jkl} v^l + \alpha_k v^i_{,j} = 0$, …(1.13)

Where $\alpha_k = \alpha_k / \alpha$ is necessary and sufficient (Takano [4] , 1966)

In fact $\alpha_k \neq 0$, there exist a suitable vector η^k, such that $\alpha_k \eta^k = 1$, then by transvection of η^k, from the condition (1.13), we have $V^i_{,j} = R^i_{jkl} v^K \eta^l$.

So, we can take concretely $f^{kl} = v^k \eta^l$.Hence, to have the concrete from f^{kl}, (1.13) should be taken as a basic condition . If this is done , we shall have (1.12) always. So, $£_v R^i_{jkl} = 0$ holds good. Thus we have the following

Theorem. If we introduce $v^i_{,j}$ by (1.13) then $£_v R^i_{jkl} = 0$ is identically satisfied

2. **Affine Motion in Tachibana Recurrent Spaces.** Firstly, we shall show the existence of affine motion in a special S-*T^c_n space of the first kind.

Differentiating (1.12) covariently with respect to x^a and using (1.5) and $A_{kl,a} = \lambda a A_{kl}$ we have

$R^i_{jkl} \alpha_{,a} = A_{kl} v^i_{,ja}$ …(2.1)

Multiplying the above equations by summing over l, we obtain

. $R^i_{jkl} v^l \alpha_{,a} = - \alpha_{,k.} v^i_{,ja}$ ...(2.2)

Where we have used

$A_{ab} v^b + \alpha_{,a} = 0$.

By virtue of (1.13), we obtain

. $R^i_{jkl} v^l = - \alpha_{,k.} v^i_{,j}$ …(2.3)

Making use of (2.3) and (2.2), we have

$\alpha_a \alpha_{,k} v^i_{,j} = \alpha_k v^i_{,ja}$ …(2.4)

since $\alpha \neq$ constant, we get

$\alpha_a . v^i_{,j} = v^i_{,ja.}$ …(2.5)

Hence (2.3) and (2.5) yield

$V^i_{.jk} + R^i_{jkl} v^l = \alpha_k v^i_{,j} - \alpha_{,k.} v^i_{,j} = 0$,

Thus, we have $£_v \Gamma^i_{jk} = 0$.

Theorem 1. An $*T^c_n$ space, satisfying $£_v\lambda_a = 0$, $\lambda_a v^a \neq 0$ constant and having resolved curvature tensor R^i_{jkl} of the form (1.13), admits naturally an affine motion.

Proof. Consider space of the second kind satisfying

$\alpha = \lambda_a v^a = 0.$

From second Bianchi identity, we have

$$\lambda_k R^i_{jla} v^a = \lambda_l R^i_{jka} v^a, \quad \ldots(2.6)$$

From where , taking care of $\lambda_l \neq 0$, we can put

$$R^i_{jkl} v^l = A^i_j \lambda_k \quad \ldots(2.7)$$

Since $\lambda_l \neq 0$, there exist suitable vector η^l, such that

$\lambda_a \eta^a = \Gamma^l$

multiplying (2.7) by η^k, we obtain

$$R^i_{jkl} \eta^k v^l = A^i_j \quad \ldots(2.8)$$

Now, introducing a non-symmetric tensor f^{xl}, which has been considered earlier in (2.8), we get

$$-R^i_{jkl} f^{kl} = -A^i_j \quad \ldots(2.9)$$

i.e., we can put

$v^i_{,j} = -A^i_j$

consequently, (2.7) may be written as

$$R^i_{jkl} v^i = -\lambda_k v^i_{,j} \quad \ldots(2.10)$$

Here, we see that

$$£_v \Gamma^i_{jk} = v^i_{,jk} - \lambda_k v^i_{,j} \quad \ldots(2.11)$$

Therefore,

$£_v \Gamma^i_{jk} = 0,$

If and only if $v^i_{,jk}$ denote a recurrence tensor with respect to the gradient recurrence vector.

Thus by the above reason, we establish

Theorem 2. An $*T^c_n$ space defined by a gradient recurrence vector λ_a and gradient characterized by $£_v \lambda_a = 0$ and $\lambda_a v^a = 0$ admits an affine motion, if and only if, the space has recurrence tensor $v^i_{,j}$ with respect of λ_k.

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