A COMPREHENSIVE REVIEW OF VARIOUS VARIATIONAL INEQUALITIES AND THEIR SYSTEMS

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Abstract: We took for the following results $u \in E, x \in B(u), y \in C(u), z \in D(u), v \in F(u)$ and $w \in G(u)$ is the solution of problem (5.1) if and only if (u, x, y, z, v, w) satisfies the relation $g(u) = m(w) + R^{\rho,A}_{\eta,M(\cdot,u)}\left[A\big(g(u) - m(w)\big) - \rho\big(N(x,y) - W(z,v) + m(w)\big)\right]$
Where $R^{\rho,A}_{\eta,M(\cdot,u)} = \big(A + \rho M(\cdot,u)\big)^{-1}$ and $\rho \in \left(0, \frac{r}{m}\right)$ is a constant and Let E be a q-uniformly smooth Banach space and $\eta: E \times E \to E$ be Lipschitz continuous mapping with constant $\mathcal{T}$. Let $A: E \to E$ be r-strongly η-accretive and Lipschitz continuous mapping with constant $\lambda_A, m: E \to E$ be Lipschitz continuous mapping with constant λ_m and $M: E \times E \to 2^E$ be (A, η)-accretive mapping in the first argument such that $g(u) - m(w) \in \mathrm{dom}\big(\mathrm{M}(\cdot, \mathrm{u})\big)$, for all $u, w \in E$. Suppose $N, W: E \times E \to E$ be Lipschitz continuous mappings in both arguments with constants $\lambda_{N_1}, \lambda_{N_2}, \lambda_{W_1}$ and λ_{W_2} , respectively and B, C, D, F and $G: E \to CB(E)$ be $\mathcal{H}$-Lipschitz continuous mappings with constants $\alpha, \beta, \gamma, \mu$ and δ, respectively. Let $g: E \to E$ be (b, ξ)-relaxed cocoercive, Lipschitz continuous mapping with constant λ_g and strongly accretive with constant l.

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1. INTRODUCTION

Because of the applications of functional analysis in sciences, engineering and social sciences, a great deal of work has been done in this area. Specially, the nonlinear analysis, a branch of functional analysis, has grown very rapidly and has many interesting applications in partial differential equations, mechanics, optimization, game theory, economics, engineering sciences etc.. The theory of variational inequalities is one of the fields of applications of non-linear analysis. It was introduced in early sixties by the Italian and French school as a joint efforts of two leading mathematicians of that period, Guido Stampacchia and Jacques-Louis Lions. This theory has many applications in different branches of science. In the last five decades, variational inequalities have been extended and generalized in different directions. A considerable interest has been shown in developing various extensions and generalizations of variational inequalities related to set-valued operators, nonconvex optimization and structural analysis. This theory was developed simultaneously not only to study the fundamental facts about the qualitative behavior of solutions of nonlinear problems, but also to solve them more efficiently numerically. In second section, we present some definitions and results from functional analysis which will be used in the sequel. The third section deals with brief introduction of variational inequalities and their generalizations. Section for is devoted to the study of system of variational inequalities (inclusions).

2. SOME BASIC CONCEPTS AND RESULTS

In this section, we present some basic notation, definitions and results of functional analysis which will be used in the subsequent chapters.

Throughout this thesis, unless otherwise specified, we assume that E is a real Banach space endowed with the norm $\|\cdot\|$, E^* is the topological dual of $E\langle\cdot,\cdot\rangle$ is the duality pairing between E and E^*, d is the metric induced by the norm $\|\cdot\|$, $CB(E)$ is the family of all closed and bounded subsets of E, 2^E is the family of all nonempty subsets of E, $\mathcal{H}(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$\mathcal{H}(P,Q) = \max\left\{\sup_{x\in P} d(x,Q), \sup_{y\in Q} d(P,y)\right\}$$

Where $d(x,Q) = \inf_{y\in Q} d(x,y)$ *and* $d(P,y) = \inf_{x\in P} d(x,y)$. We denote by X a real Hilbert space unless otherwise specified.

Definition : 2.1

A continuous and strictly increasing function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$ is called a gauge function.

Definition : 2.2

Given a gauge function φ, the mapping $\mathcal{T}_\varphi: E \to 2^{E^*}$ defined by

$$\mathcal{T}_\varphi(x) = \{f \in E^*: \langle x,f\rangle = \|x\|\|f\|; \|f\| = \varphi(\|x\|)\}, \text{for all } x \in E,$$

is called the duality mapping with gauge function φ.

Definition: 2.3

Let E be real Banach space. Then

(i) a mapping $\mathcal{T}: E \to 2^{E^*}$ is called normalized duality mapping defined by

$$\mathcal{T}(x) = \{f \in E^*: \langle x,f\rangle = \|x\|\|f\|; \|x\| = \|f\|\}, \text{for all } x \in E.$$

(ii) a mapping $\mathcal{T}_q: E \to 2^{E^*}, q > 1$ is called generalized duality mapping defined by

$$\mathcal{T}_q(x) = \{f \in E^*: \langle x,f\rangle = \|x\|^q; \|f\| = \|x\|^{q-1}\}, \text{for all } x \in E.$$

Remark: 2.1

In particular, for $q = 2$ the generalized duality mapping coincides with the normalized duality mapping.

Definition: 2.4

A set-valued mapping $T: E \to CB(E)$ is said to be $\mathcal{H}$-Lipschitz continuous, if there exists a constant $\xi_1 > 0$ such that

$$\mathcal{H}(Tx, Ty) \le \xi_1\|x - y\|, \text{for all } x, y \in E.$$

Theorem: 2.1

Let (E, d) be a metric space. If $F: X \to CB(E)$ is a set-valued contraction mapping, then F has fixed point.

Definition: 2.5

A Banach space E is said to be uniformly convex, if for any given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E, \|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| = \epsilon$, we have $\|x + y\| \leq 2(1 - \delta)$. The function $\delta_{E(\epsilon)} = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = 1, \|y\| = 1, \|x - y\| = \epsilon\right\}$

is called the modulus of convexity of the space E.

Definition: 2.6

The modulus of smoothness of E is the function $\rho E: [0, \infty) \to [0, \infty)$ defined by

$$\rho_{E(\epsilon)} = \sup\left\{1 - \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t\right\}$$

A Banach space E is called uniformly smooth, if $\lim_{t\to 0} \frac{\rho_E(t)}{t}$.

Definition: 2.7

The Banach space E is called q-uniformly smooth, if there exists a constant $C > 0$ such that $\rho_E(t) \leq Ct^q, q > 1$

Proposition: 2.1

Let E be a real Banach space and $\mathcal{T}: E \to 2^{E^*}$ be a normalized duality mapping. Then for any $x, y \in E, j(x + y) \in \mathcal{T}(x + y)$

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$;

(ii) $\langle x - y, j(x) - j(y)\rangle \leq 2D^2_{\rho_E}\left(\frac{4\|x-y\|}{D}\right)$, where $\mathrm{D} = \sqrt{\|x\|^2 + \|y\|^2/2}$.

Lemma: 2.1

Let E be a real uniformly smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant $C_q > 0$ such that, for all $x, y \in E$,

$$\|x + y\|^q \leq \|x\|^q + q\langle y, \mathcal{T}_q(x)\rangle + C_q\|y\|^q$$

Definition: 2.8

Let $\varphi: E \to \mathbb{R} \cup \{+\infty\}$ be a proper function, φ is said to be subdifferential at a point $x \in E$, if there exists a point $f \in E^*$ such that $\varphi(y) - \varphi(x) \geq \langle f, y - x\rangle$, for all $y \in E$,

Where f is called subgradient of φ at x. The set of all subgradient of φ at x is denoted by $\partial_\varphi(x)$.

The mapping $\partial_\varphi: E \to 2^{E^*}$defined by $\partial_\varphi(x) = \{f \in E^*: \varphi(y) - \varphi(x) \geq \langle f, y - x\rangle, \text{for all } y \in E\}$

is called subdifferentail of φ at x.

Definition: 2.9

Let X be a real Hilbert space and $g: X \to X$ be a single-valued mapping. Then g is said to be

(i) monotone, if

$$\langle g(x) - g(y), x - y\rangle \geq 0, \text{for all } x, y \in X;$$

(ii) strictly monotone, if
$$\langle g(x) - g(y), x - y\rangle > 0, \text{for all } x, y \in X,$$
(iii) strongly monotone, if there exists a constant $\xi_2 > 0$ such that
$$\langle g(x) - g(y), x - y\rangle \geq \xi_2 \|x - y\|^2, \text{for all } x, y \in X;$$
(iv) Lipschitz continuous, if there exists a constant $\lambda_g > 0$ such that
$$\|g(x) - g(y)\| \leq \lambda_g \|x - y\|, \text{for all } x, y \in X;$$
(v) α_1-expansive, if there exists a constant $\alpha_1 > 0$ such that
$$\|g(x) - g(y)\| \geq \alpha_1 \|x - y\|, \text{for all } x, y \in X;$$
If $\alpha_1 = 1$, then it is expansive.

Definition: 2.10

Let E be a real Banach space. Let $g: E \to E$ is a single-valued mapping. Then g is said to be

(i) accretive, if for any $x, y \in E$, there exists $j(x - y) \in \mathcal{T}(x - y)$ such that
$$\langle g(x) - g(y), j(x - y)\rangle \geq 0;$$
(ii) strictly accretive, if for any $x, y \in E$, there exists $j(x - y) \in \mathcal{T}(x - y)$ such that
$$\langle g(x) - g(y), j(x - y)\rangle > 0;$$
and equality holds if and only if $x = y$;

(iii) k_1- strongly accretive, if for any $x, y \in E$,there exists $j(x - y) \in \mathcal{T}(x - y)$ and a constant $k_1 > 0$ such that
$$\langle g(x) - g(y), j(x - y)\rangle \geq k_1 \|x - y\|^2;$$
(iv) k_1- strongly accretive, if for any $x, y \in E$,there exists $j(x - y) \in \mathcal{T}(x - y)$ and a constant $k_1 > 0$ such that
$$\langle g(x) - g(y), j(x - y)\rangle \geq k_1 \|x - y\|^2;$$
(v) m_1- relaxed accretive, if for any $x, y \in E$,there exists $j(x - y) \in \mathcal{T}(x - y)$ and a constant $m_1 > 0$ such that
$$\langle g(x) - g(y), j(x - y)\rangle \geq -m_1 \|x - y\|^2.$$

3. VARIATIONAL INEQUALITIES

Many problems of elasticity and fluid mechanics can be expressed in terms of a n unknown u, representing the displacement of a mechanical system, satisfying

$$a(u, v - u) \geq F\,(v - u), \text{for all } v \in K, \tag{3.1}$$

where K is a nonempty, closed, convex subset of a Hilbert space, $X, a(\cdot,\cdot)$ is a bilinear from and F is a bounder linear functional on X. The inequalities of the type (3.1) are called variational inequalities. If the bilinear from $a(\cdot,\cdot)$ is continuous, then by Reiz-representation theorem, we have

$$a(u, v) = \langle A(u), v\rangle, \text{for all } u, v \in X, \tag{3.2}$$

Where A is a continuous linear operator on X. Then equality (3.1) is equivalent to find $u \in K$ such that

$$\langle A(u), v - u\rangle \geq \langle F, v - u\rangle, \text{for all } v \in K \tag{3.3}$$

If the operators A and F are nonlinear, then the variational inequality (3.3) is known as strongly nonlinear variation inequality, introduced and studied by many authors. If $F = 0$, then (3.3) is equivalent to find $u \in K$ such that

$$\langle A(u), v - u\rangle \geq 0, \text{for all } v \in K. \tag{3.4}$$

The variational inequality of the type (3.4) was introduced and studied by Fichera in 1964. Lion and Stampacchia proved the existence of unique solution of (3.4) using the projection techniques. Variational-like inequality is a generalization of variational inequality, which is introduced and studied by Parida, Sahoo and Kumar. Let K be a closed convex set in $\mathbb{R}^n$. Given two continuous mappings $f: K \to \mathbb{R}^n$ and $\eta: K \times K \to \mathbb{R}^n$, then the variational-like inequality problem is to find $u \in K$ such that

$$\langle f(u), \eta(u,v) \rangle \geq 0, \text{for all } v \in K. \tag{3.5}$$

Remark: 3.1

If $\eta(u,v) = u - v$, then variational-like inequality (3.5) is equivalent to the variational inequality (3.4). Dien introduced and studied the following general variational-like inequality problem in $\mathbb{R}^n$. Given $\phi: K \to \mathbb{R}$, find $u \in K$ such that

$$\langle f(u), \eta(u,v) \rangle \geq \phi(u) - \phi(v), \text{for all } v \in K. \tag{3.6}$$

It has been further studied by Siddiqi, Ansari and Ahmad in the setting of reflexive Banach spaces and topological vector spaces with or without convexity assumptions. In many applications, the convex set in the formulation of variational-like inequality problem also depends upon the solution itself. In this case variational-like inequality problem is called quasi-variational-like inequality problem. More precisely, for a given set-valued mapping $Q: K \to 2^K$, the general quasi-variational-like inequality problem is the following:

Find $u \in K$ such that $u \in Q(u)$ and $\langle f(u), \eta(u,v) \rangle \geq \phi(u) - \phi(v), \text{for all } v \in Q(u)$. (3.7).
Where f, η and ϕ are same in (3.6).

In 1994, Hassouni and Moudafi used the resolvent operator technique for maximal monot one mappings to study mixed type variational inequalities with single-valued mappings, which are called variational inclusions and developed a perturbed algorithm for finding approximate solutions of mixed variational inequalities. Given continuous mappings $T, g: X \to X$ with $Im\,(g) \cap \mathrm{dom}(\partial\varphi) \neq \emptyset$. Then the following problem of finding $u \in X$ such that $g(u) \cap \mathrm{dom}(\partial\varphi) \neq \emptyset$ and

$$\langle T(u) - A(u), v - g(u) \rangle \geq \varphi\big(g(u)\big) - \varphi(v), \text{for all } v \in X \tag{3.8}$$

Where A is a nonlinear continuous mapping on X, $\partial\varphi$ denotes the subdifferential of a proper, convex and lower-semicontinuous function $\varphi: X \to \mathbb{R} \cup \{\infty\}$, $\mathrm{dom}(\partial\varphi)$ denotes the domain of $\partial\varphi$. Problem (3.8) is called variational inclusion problem, introduced and studied. If $\varphi \equiv \delta_K$, the indicator function of a closed convex set K in H defined by

$$\delta_K(x) = \begin{cases} 0, & x \in K \\ +\infty, & \text{otherwise} \end{cases}$$

Then the variational inclusion problem reduces to the following strongly non-linear variational inequality problem. $\langle T(u) - A(u), v - g(u) \rangle \geq 0, \text{for all } v \in K$ (3.9)

4. SYSTEM OF VARIATIONAL INEQUALITIES

In recent past, system of inequalities are used as tools to solve various equilibrium problems e.g., Nash equilibrium problem, spatial equilibrium problem and general equilibrium programming problem, problems from operations research, economics, game theory, mathematical physics and other areas, see for example and references therein. Pang uniformly modeled these equilibrium problems in the form of a variational inequality defined on a product of sets. He decomposed the original variational inequality into a system of variational inequalities, which are easy to solve, to establish some solution methods for finding the approximate solutions of above mentioned equilibrium problems. Later, it is found that these two problems, variational inequality defined on a product of sets and system of variational inequalities, are equivalent. Kassay and Kolumban introduced the following system of variational inequalities and proved the existence of solutions using Ky-Fan's lemma. Let X_1 and X_2 are two Hilbert spaces, $A \subset X_1$ and $B \subset X_2$ are two nonempty closed and convex sets. Let $F: X_1 \times X_2 \to X_1, G: X_1 \times X_2 \to X_2$ be the single-valued mappings. Find $(a,b) \in A \times B$ such that

$$\begin{cases} \langle F(a,b), x-a \rangle \geq 0, & \text{for all } x \in A \\ \langle G(a,b), y-b \rangle \geq 0, & \text{for all } y \in B \end{cases} \qquad (4.1)$$

Verma introduced and studied the following system of nonlinear variational inequalities.

Let X be a real Hilbert space and A be a closed, convex subset of X. $T: A \to X$ is a nonlinear mapping and $\rho, \gamma > 0$ are constants. Find $(a,b) \in A \times A$ such that

$$\begin{cases} \langle \rho T(b) + a - b, x-a \rangle \geq 0, & \text{for all } x \in A \\ \langle \gamma T(b) + b - a, x-b \rangle \geq 0, & \text{for all } y \in A \end{cases} \qquad (4.2)$$

Lan introduced a following system of A-monotone set-valued variational inclusions in Hilbert spaces. Let X_1 and X_2 be two real Hilbert spaces, $S: X_1 \times X_2 \to X_1, T: X_1 \times X_2 \to X_2$, $p: X_1 \to X_1$ and $q: X_2 \to X_2$ be the single-valued mappings, $F: X_1 \to 2^{X_1}$, $G: X_2 \to 2^{X_2}$ be the set-valued mappings. Let $A_1: X_1 \to X_1$ and $A_2: X_2 \to X_2, M: X_1 \to 2^{X_1}, N: X_2 \to 2^{X_2}$ be any nonlinear mappings and $f: X_1 \to X_1, g: X_2 \to X_2$ be nonlinear mappings with $f(X_1) \cap \mathrm{dom}(M) = \emptyset$ and $g(X_2) \cap \mathrm{dom}(N) \neq \emptyset$ respectively. Find $(x,y) \in X_1 \times X_2, u \in F(x), v \in G(y)$ such that

$$\begin{cases} 0 \in S(p(x), v) + M(f(x)) \\ 0 \in T(u, q(y)) + N(g(x)) \end{cases} \qquad (4.3)$$

Recently , Zou and Huang introduced and study the following system of variational inclusions involving $H(\cdot,\cdot)$-accretive operator in q-uniformly smooth Banach spaces. Let E_1 and E_2 are two q-uniformly smooth Banach spaces and $C_1 \subset E_1$ and $C_2 \subset E_2$ are two nonempty, closed and convex sets. Let $F: E_1 \times E_2 \to E_1, G: E_1 \times E_2 \to E_2$, $H_1: E_1 \times E_1 \to E_1, H_2: E_2 \times E_2 \to E_2$, $A_1, B_1: E_1 \to E_1$, $A_2, B_2: E_2 \to E_2$ be the single-valued mappings. Furthermore, let $M: E_1 \to 2^{E_1}$ be set-valued $H_1(A_1, B_1)$-accretive operator and $N: E_2 \to 2^{E_2}$ be set-valued $H_2(A_2, B_2)$-accretive operator. Find $(a,b) \in E_1 \times E_2$, such that

$$\begin{cases} 0 \in F(a,b) + M(a) \\ 0 \in G(a,b) + N(b) \end{cases} \qquad (4.4)$$

The following system of variation inequality (inclusion) can be obtained from the above system.

(i) If X_1, X_2 are two Hilbert spaces, $H_1(A_1, B_1) = H_1, H_2(A_2, B_2) = H_2$, M is (H_1, η)-monotone and N is also (H_2, η)-monotone, then the problem (4.4) becomes the system of variational inclusions studied by Fang, Huang and Thompson.

(ii) If X_1 and X_2 are two Hilbert spaces, $H_1(A_1, B_1) = A_1, H_2(A_2, B_2) = A_2$ and $M(x) = \partial\phi(x), N(y) = \partial\psi(y)$, for all $(x,y) \in X_1 \times X_2$, where $\phi: X_1 \to \mathbb{R} \cup \{\infty\}$ and $\psi: X_2 \to \mathbb{R} \cup \{\infty\}$ are two proper, convex, lower semicontinuous functional and $\partial\phi$ and $\partial\psi$ denotes the subdifferential operator of ϕ and ψ, respectively. Then the problem (4.4) becomes the following problem. Find $(a,b) \in C_1 \times C_2$ such that

$$\begin{cases} \langle F(a,b), x-a \rangle + \phi(x) - \phi(a) \geq 0 & \text{for all } x \in X_1 \\ \langle G(a,b), y-b \rangle + \psi(y) - \psi(b) \geq 0 & \text{for all } y \in X_2 \end{cases} \qquad (4.5)$$

which is called a system of nonlinear variational inequalities.

(iii) If X_1, X_2 are two Hilbert spaces, $H_1(A_1, B_1) = A_1, H_2(A_2, B_2) = A_2$ and $M(x) = \partial\delta_{C_1}(x), N(y) = \partial\delta_{C_2}(y)$, for all $(x,y) \in C_1 \times C_2$, where $C_1 \subset X_1$ and $C_2 \subset X_2$ are nonempty, closed, convex, $\partial\delta_{C_1}$ and $\partial\delta_{C_2}$ denote the indicator functions of C_1 and C_2, respectively inequalities studied .

(iv) If $X_1 = X_2 = X$, $C_1, C_2 = C$ and $F(x,y) = \rho T(y) + x - y, G(x,y) = \gamma T(x) + y - x$, for all $x, y \in X$, where $T: C \to X$ is a nonlinear mapping $\rho > 0, \gamma > 0$ are two constants, then the problem (4.1) reduces to the following system of variational inequalities. Find $(a,b) \in C \times C$ such that

$$\begin{cases} \langle \rho \ T(b) + a - b, x - a \ \rangle \geq 0 & \text{for all } x \in C \\ \langle \gamma T(a) + b - a, y - b \ \rangle \geq 0 & \text{for all } y \in C \end{cases} \qquad (4.6)$$

which is the system of nonlinear variational inequalities considered by Verma.

5. GENERALIZED QUASI-VARIATIONAL-LIKE INCLUSIONS WITH (A,η)-ACCRETIVE AND RELAXED COCOERCIVE MAPPINGS

Throughout this section, we take E to be q-uniformly smooth Banach space. This section is devoted to study a generalized quasi-variational-like inclusion problem involving (A,η)-accretjve and relaxed cocoercive mappings. An iterative algorithm is constructed to approximate the solutions of generalized quasi-variational-like inclusion problem. Finally, some applications are given. Let $N, W, \eta: E \times E \to E, g, m, A: E \to E$ be the single-valued mappings, $B, C, D, F, G: E \to 2^E$ be the set-valued mappings. Let $M: E \times E \to 2^E$ be an (A,η)-accretive mapping in the first argument such that $g(u) - m(w) \in \mathrm{dom}\big(M(\cdot,u)\big)$, for all $u, w \in E$. We consider the following generalized quasi-variational-like inclusion problem. Find $u \in E, x \in B(u), y \in C(u), z \in D(u), v \in F(u)$ and $w \in G(u)$ such that

$$0 \in N(x,y) - W(z,v) + m(w) + M(g(u) - m(w), u) \qquad (5.1)$$

Below are some special cases of problem (5.1).

Some special cases:

(i) If $m = 0, M(g(u) - m(w), u) = M\big(g(u)\big)$ and $W, D, F = 0$, then problem (5.1) reduces to the problem of finding $u \in E, x \in B(u), y \in C(u)$ such that

$$0 \in N(x,y) + M\big(g(u)\big) \qquad (5.2)$$

Problem (5.2) is considered by Chang.

(ii) If B and C are single-valued mappings, then problem (5.2) can be replaced by finding $u \in E$ such that

$$0 \in N\big(B(u), C(u)\big) + M\big(g(u)\big) \qquad (5.3)$$

A problem similar to problem (5.3) is considered by Lan.

(iii) If $C = 0$ and $B, g = I$, the identity mappings, the (5.3) reduces to the problem of finding $u \in E$ such that

$$0 \in N(u) + M(u) \qquad (5.4)$$

which is considered by Bi et al.

Lemma: 5.1

$u \in E, x \in B(u), y \in C(u), z \in D(u), v \in F(u)$ and $w \in G(u)$ is the solution of problem (5.1) if and only if (u, x, y, z, v, w) satisfies the relation

$$g(u) = m(w) + R^{\rho,A}_{\eta,M(\cdot,u)}\Big[A\big(g(u) - m(w)\big) - \rho\big(N(x,y) - W(z,v) + m(w)\big)\Big] \qquad (5.5)$$

Where $R^{\rho,A}_{\eta,M(\cdot,u)} = \big(A + \rho M(\cdot,u)\big)^{-1}$ and $\rho \in \left(0, \frac{r}{m}\right)$ is a constant.

Proof:

The proof follow directly from the known definition.

Algorithm: 5.1

For any given $u_0 \in E$, we choose $x_0 \in B(u_0), y_0 \in C(u_0), z_0 \in D(u_0), v_0 \in F(u_0), w_0 \in G(u_0)$ and $\{u_n\}, \{x_n\}, \{y_n\}, \{z_n\}, \{v_n\}$ and $\{w_n\}$ by the following iterative schemes.

$$g(u_{n+1}) = m(w_n) + R^{\rho,A}_{\eta,M(\cdot,u_n)}\left[A\big(g(u_n) - m(w_n)\big) - \rho\big(N(x_n,y_n) - W(z_n,v_n) + m(w_n)\big)\right] \quad (5.6)$$

$$x_{n+1} \in B(u_{n+1}), \|x_{n+1} - x_n\| \le \mathcal{H}\big(B(u_{n+1}), B(u_n)\big); \quad (5.7)$$

$$y_{n+1} \in C(u_{n+1}), \|y_{n+1} - y_n\| \le \mathcal{H}\big(C(u_{n+1}), C(u_n)\big); \quad (5.8)$$

$$z_{n+1} \in D(u_{n+1}), \|z_{n+1} - z_n\| \le \mathcal{H}\big(D(u_{n+1}), D(u_n)\big); \quad (5.9)$$

$$v_{n+1} \in F(u_{n+1}), \|v_{n+1} - v_n\| \le \mathcal{H}\big(F(u_{n+1}), F(u_n)\big); \quad (5.10)$$

$$w_{n+1} \in G(u_{n+1}), \|w_{n+1} - w_n\| \le \mathcal{H}\big(G(u_{n+1}), G(u_n)\big); \quad (5.11)$$

Where $n = 0,1,2,3, \dots.., \rho \in \left(0, \frac{r}{m}\right)$ is a constant. Now, we prove the following existence and convergence result for generalized quasi-variational-like inclusion problem.

Theorem: 5.1

Let E be a q-uniformly smooth Banach space and $\eta: E \times E \to E$ be Lipschitz continuous mapping with constant $\mathcal{T}$. Let $A: E \to E$ be r-strongly η-accretive and Lipschitz continuous mapping with constant $\lambda_A, m: E \to E$ be Lipschitz continuous mapping with constant λ_m and $M: E \times E \to 2^E$ be (A,η)-accretive mapping in the first argument such that $g(u) - m(w) \in \mathrm{dom}\big(\mathrm{M}(\cdot,\mathrm{u})\big)$, for all $u, w \in E$. Suppose $N, W: E \times E \to E$ be Lipschitz continuous mappings in both arguments with constants $\lambda_{N_1}, \lambda_{N_2}, \lambda_{W_1}$ and λ_{W_2} , respectively and B, C, D, F and $G: E \to CB(E)$ be $\mathcal{H}$-Lipschitz continuous mappings with constants $\alpha, \beta, \gamma, \mu$ and δ, respectively. Let $g: E \to E$ be (b,ξ)-relaxed cocoercive, Lipschitz continuous mapping with constant λ_g and strongly accretive with constant l.

Suppose that there exists $\rho \in \left(0, \frac{r}{m}\right)$ and $t > 0$ such that the following conditions hold.

$$\left\| R^{\rho,A}_{\eta,M0(\cdot,u_n)}(x) - R^{\rho,A}_{\eta,M(\cdot,u_{n-1})}(x) \right\| \le t\|u_n - u_{n-1}\|, \text{for all } u_n, u_{n-1} \in E \quad (5.12)$$

and

$$0 < \lambda_m \delta(\rho + \lambda_A) + \lambda_A \sqrt[q]{(1 - q\xi + \big(qp + C_q\big)\lambda_G^q} + \rho \sqrt[q]{\big(\lambda_{N_1}\alpha + \lambda_{N_2}\beta\big)^q - \big(q - C_q\big)\big(\lambda_{W_1}\gamma + \lambda_{W_2}\mu\big)^q} \quad (5.13)$$
$$+\lambda_A < \frac{[l - \lambda_m\delta + t](r - \rho m)}{\gamma^{q-1}}, l > \lambda_m\delta + t)$$

Where C_q is the constant as in Lemma, then the iterative sequences $\{u_n\}, \{x_n\}, \{y_n\}, \{z_n\}, \{v_n\}$ and $\{w_n\}$ generated by Algorithm (5.1) converge strongly to u, x, y, z, v and w, respectively and (u, x, y, z, v, w) is a solution of problem.

Proof:

From Algorithm 5.1, Proposition, we have

$$\begin{aligned}
&\|g(u_{n+1})-g(u_n)\| \\
&\qquad = \Big\| m(w_n) + R^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A\big(g(u_n)-m(w_n)\big)-\rho\big(N(x_n,y_n)-W(z_n,v_n)+m(w_n)\big)\big] \\
&\qquad - \Big\{ m(w_{n-1}) \\
&\qquad + R^{\rho,A}_{\eta,M(\cdot,u_{n-1})}\big[A\big(g(u_{n-1})-m(w_{n-1})\big) \\
&\qquad - \rho\big(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1})+m(w_{n-1})\big)\big]\Big\}\Big\|
\end{aligned}$$

$$\begin{aligned}
&\le \|m(w_n)-m(w_{n-1})\| \\
&\quad + \Big\| R^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A\big(g(u_n)-m(w_n)\big)-\rho\big(N(x_n,y_n)-W(z_n,v_n)+m(w_n)\big)\big] \\
&\quad - R^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A\big(g(u_{n-1})-m(w_{n-1})\big)-\rho\big(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1})+m(w_{n-1})\big)\big]\Big\| \\
&\quad + \Big\| R^{\rho,A}_{\eta,M(\cdot,u_{n-1})}\big[A\big(g(u_{n-1})-m(w_{n-1})\big)-\rho\big(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1})+m(w_{n-1})\big)\big] \\
&\quad - R^{\rho,A}_{\eta,M(\cdot,u_n)}\big[A\big(g(u_{n-1})-m(w_{n-1})\big)-\rho\big(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1})+m(w_{n-1})\big)\big]\Big\|
\end{aligned}$$

$$\begin{aligned}
&\le \|m(w_n)-m(w_{n-1})\| \\
&\quad + \frac{t^{q-1}}{r-\rho m}\big[\big\|A\big(g(u_n)-m(w_n)\big)-A\big(g(u_{n-1})-m(w_{n-1})\big) \\
&\quad - \rho\{\big(N(x_n,y_n)-W(z_n,v_n)+m(w_n)\big)-\big(N(x_{n-1},y_{n-1})-W(z_{n-1},v_{n-1})+m(w_{n-1})\big)\big\|\big] \\
&\quad + t\|u_n-u_{n-1}\|
\end{aligned}$$

$$\begin{aligned}
&\le \left(1+\frac{t^{q-1}}{r-\rho m}\right)\|m(w_n)-m(w_{n-1})\| + \frac{t^{q-1}}{r-\rho m}\big\|A\big(g(u_n)-m(w_n)\big)-A\big(g(u_{n-1})-m(w_{n-1})\big)\big\| \\
&\quad + \frac{t^{q-1}}{r-\rho m}\rho\big\|N(x_n,y_n)-N(x_{n-1},y_{n-1})-\big(W(z_n,v_n)\big)-W(z_{n-1},v_{n-1})\big\| \\
&\quad + t\|u_n-u_{n-1}\| \qquad (5.14)
\end{aligned}$$

Since A is λ_A-Lipschiyz continuous, we have

$$\begin{aligned}
&\|g(u_{n+1})-g(u_n)\| \\
&\qquad \le \left[1+\frac{t^{q-1}}{r-\rho m}(\rho+\lambda_A)\right]\|m(w_n)-m(w_{n-1})\| \\
&\qquad + \frac{t^{q-1}}{r-\rho m}\lambda_A\big\|u_n-u_{n-1}-\big(g(u_n)-g(u_{n-1})\big)\big\| \\
&\qquad + \frac{t^{q-1}}{r-\rho m}\rho\big\|N(x_n,y_n)-N(x_{n-1},y_{n-1})-\big(W(z_n,v_n)\big)-W(z_{n-1},v_{n-1})\big\| \\
&\qquad + \left(\frac{t^{q-1}}{r-\rho m}\lambda_A+t\right)\|\,u_n-u_{n-1}\| \qquad (5.15)
\end{aligned}$$

Since m is Lipschitz continuous with constant λ_m and G is $\mathcal{H}$ –Lipschits continuous with constant δ, we have

$$\begin{aligned}
\|m(w_n)-m(w_{n-1})\| &\le \lambda_m\|w_n-w_{n-1}\| \le \lambda_m\mathcal{H}\big(G(u_n)-G(u_{n-1})\big) \\
&\le \lambda_m\delta\|\,u_n-u_{n-1}\| \qquad (5.16)
\end{aligned}$$

Since g is (b,ξ)-relaxed cocoercive and λ_g-Lipschitz continuous, we have

$$\big\|u_n-u_{n-1}-\big(g(u_n)-g(u_{n-1})\big)\big\|^{\boldsymbol{q}}$$

$$\le \|u_n-u_{n-1}\|^{\boldsymbol{q}} - q\langle g(u_n)-g(u_{n-1}),\mathcal{J}_q(u_n-u_{n-1})\rangle + C_q\big\|\big(g(u_n)-g(u_{n-1})\big)\big\|^{\boldsymbol{q}}$$

$$\leq \|u_n - u_{n-1}\|^q + qb\left\|\left(g(u_n) - g(u_{n-1})\right)\right\|^q - q\xi\|u_n - u_{n-1}\|^q + C_q\lambda_g^q\|u_n - u_{n-1}\|^q$$

$$\leq \|u_n - u_{n-1}\|^q + qb\lambda_g^q\|u_n - u_{n-1}\|^q - q\xi\|u_n - u_{n-1}\|^q + C_q\lambda_g^q\|u_n - u_{n-1}\|^q$$

$$= \left(1 - q\xi + \left(qb + C_q\right)\lambda_g^q\right)\|u_n - u_{n-1}\|^q$$

Thus, we have

$$\left\|u_n - u_{n-1} - \left(g(u_n) - g(u_{n-1})\right)\right\| \leq \sqrt{1 - q\xi + \left(qb + C_q\right)\lambda_g^q}\,\| u_n - u_{n-1}\| \tag{5.17}$$

Also

$$\begin{aligned}\left\|N(x_n, y_n) - N(x_{n-1}, y_{n-1}) - \left(W(z_n, v_n) - W(z_{n-1}, v_{n-1})\right)\right\|^q \\ \leq \|N(x_n, y_n) - N(x_{n-1}, y_{n-1})\|^q \\ - \left(q - C_q\right)\|W(z_n, v_n) - W(z_{n-1}, v_{n-1})\|^q\end{aligned} \tag{5.18}$$

By using Lipschitz continuity of N with constant λ_{N_1} for the first argument and λ_{N_2} for the second argument and $\mathcal{H}$-Lipschitz continuity of B and C with constants α and β, respectively, we have

$$\|N(x_n, y_n) - N(x_{n-1}, y_{n-1})\|$$

$$= \|N(x_n, y_n) - N(x_{n-1}, y_{n-1}) + N(x_n, y_{n-1}) - N(x_{n-1}, y_{n-1})\|$$

$$\leq \|N(x_n, y_n) - N(x_{n-1}, y_{n-1})\| + \|N(x_n, y_{n-1}) - N(x_{n-1}, y_{n-1})\|$$

$$\leq \lambda_{N_2}\|y_n - y_{n-1}\| + \lambda_{N_1}\|x_n - x_{n-1}\|$$

$$= \left(\lambda_{N_1}\alpha + \lambda_{N_2}\beta\right)\|u_n - u_{n-1}\| \tag{5.19}$$

Thus

$$\|N(x_n, y_n) - N(x_{n-1}, y_{n-1})\|^q \leq \left(\lambda_{N_1}\alpha + \lambda_{N_2}\beta\right)^q\|u_n - u_{n-1}\|^q \tag{5.20}$$

Using the similar arguments as for (5.19), we have

$$\|W(z_n, v_n) - W(z_{n-1}, v_{n-1})\|^q \leq \left(\lambda_{W_1}\gamma + \lambda_{W_2}\mu\right)^q\|u_n - u_{n-1}\|^q \tag{5.21}$$

Using (5.20) and (5.21), (5.18) becomes

$$\begin{aligned}\|N(x_n, y_n) - N(x_{n-1}, y_{n-1}) - [W(z_n, v_n) - W(z_{n-1}, v_{n-1})]\|^q \\ = \left[\left(\lambda_{N_1}\alpha + \lambda_{N_2}\beta\right)^q - \left(q - C_q\right)\left(\lambda_{W_1}\gamma + \lambda_{W_2}\mu\right)^q\right]\|u_n - u_{n-1}\|^q\end{aligned} \tag{5.22}$$

It follows that

$$\begin{aligned}\left\|N(x_n, y_n) - N(x_{n-1}, y_{n-1}) - \left(W(z_n, v_n) - W(z_{n-1}, v_{n-1})\right)\right\| \\ \leq \sqrt[q]{\left(\lambda_{N_1}\alpha + \lambda_{N_2}\beta\right)^q - \left(q - C_q\right)\left(\lambda_{W_1}\gamma + \lambda_{W_2}\mu\right)^q}\,\|u_n - u_{n-1}\|\end{aligned} \tag{5.23}$$

Combining (5.16), (5.17), (5.23) with (5.15), we obtain

$$\begin{aligned}\|g(u_{n+1}) - g(u_n)\| &\le \left[\lambda_m\delta + \frac{T^{q-1}\lambda_m\delta}{r-\rho m}(\rho+\lambda_A)\right]\|u_n - u_{n-1}\| \\ &+ \frac{T^{q-1}\rho}{r-\rho m}\lambda_A\sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q}\|u_n - u_{n-1}\| \\ &+ \frac{T^{q-1}\rho}{r-\rho m}\sqrt[q]{(\lambda_{N_1}\alpha + \lambda_{N_2}\beta)^q - (q - C_q)(\lambda_{W_1}\gamma + \lambda_{W_2}\mu)^q}\,\|u_n - u_{n-1}\| \\ &+ \left(\frac{T^{q-1}\lambda_A}{r-\rho m} + t\right)\|u_n - u_{n-1}\|\end{aligned}$$

$$\begin{aligned}\le \Bigg[\lambda_m\delta + \frac{T^{q-1}\lambda_m\delta}{r-\rho m}(\rho+\lambda_A) + \frac{T^{q-1}\rho}{r-\rho m}\lambda_A\sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q} \\ + \frac{T^{q-1}}{r-\rho m}\rho\sqrt[q]{(\lambda_{N_1}\alpha + \lambda_{N_2}\beta)^q - (q - C_q)(\lambda_{W_1}\gamma + \lambda_{W_2}\mu)^q} + \frac{T^{q-1}}{r-\rho m}\lambda_A + t\Bigg]\|u_n \\ - u_{n-1}\| \qquad (5.24)\end{aligned}$$

By the strong accretivity of g with constant l. We have

$$\begin{aligned}\|g(u_{n+1}) - g(u_n)\|.\|u_{n+1} - u_n\|^{q-1} &\ge \langle g(u_{n+1}) - g(u_n), \mathcal{J}_q(u_{n+1} - u_n)\rangle \\ &\ge l\|u_{n+1} - u_n\|^q\end{aligned}$$

Which implies that

$$\|u_n - u_{n-1}\| \le \frac{1}{l}\|g(u_{n+1}) - g(u_n)\| \qquad (5.25)$$

Combining (5.24) and (5.25), we have

$$\|u_{n+1} - u_n\| \le \theta\|u_n - u_{n-1}\| \qquad (5.26)$$

Where

$$\begin{aligned}\theta = \Bigg[\lambda_m\delta + \frac{T^{q-1}\lambda_m\delta}{r-\rho m}(\rho+\lambda_A) + \frac{\mathcal{T}^{q-1}\rho}{r-\rho m}\lambda_A\sqrt[q]{1 - q\xi + (qb + C_q)\lambda_g^q} \\ + \frac{T^{q-1}}{r-\rho m}\rho\sqrt[q]{(\lambda_{N_1}\alpha + \lambda_{N_2}\beta)^q - (q - C_q)(\lambda_{W_1}\gamma + \lambda_{W_2}\mu)^q} + \frac{T^{q-1}}{r-\rho m}\lambda_A + t\Bigg]/l\end{aligned}$$

By (5.13), we know that $\theta < 1$ and so (5.26) implies that $\{u_n\}$ is a Cauchy sequence. Thus, there exists $u \in E$ such that $u_n \to u$ as $n \to \infty$. By the $\mathcal{H}$-Lipschitz continuity of set-valued mappings B, C, D, F and G and (5.6)-(5.11) of Algorithm (5.1), it follows that $x_n \to x$, $y_n \to y$, $z_n \to z$, $v_n \to v$ and $w_n \to w$.

As $A, \eta, M, N, W, B, C, D, F, G, m, g$ and $R_{\eta,M}^{\rho,A}$ are all continuous and by Algorithm (5.1), it follows that u, x, y, z, v, w satisfy the following relation.

$$g(u) = m(w) + R_{\eta,M(\cdot,u_n)}^{\rho,A}\big[A\big(g(u) - m(w)\big) - \big(N(x,y) - W(z,v) + m(w)\big)\big]$$

It follows that (u, x, y, z, v, w) is a solution of generalized quasi-variational-like inclusion problem. This completes the proof.

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