EXTENSIONS OF WEAKLY ESSENTIAL SUPPLEMENTED MODULES

[1]Sushma Jat,
Asso.Professor, LNCT, Bhopal, India.
[2]Vivek Prasad Patel
Asso. Professor,
OIST, Bhopal, India.

Abstract

It is shown that weakly essential supplemented modules need not be closed under extension (i.e. if U and M/U are weakly essential supplemented then M need not be weakly essential supplemented). We prove that, if U has a weak essential supplement in M then M is weakly essential supplemented. For a commutative ring R, we prove that R is semilocal if and only if every direct product of simple R-modules is weakly essential supplemented.

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Introduction

Throughout, R is a commutative ring with identity and M is a unital left R-module. By $N \subseteq M$, we mean that N is a submodule of M. A submodule $L \subseteq M$ is said to be essential in M, denoted as $L \trianglelefteq M$, if $L \cap N \neq 0$ for every nonzero submodule $N \subseteq M$. . A submodule S of M is called small (in M), denoted as $S \ll M$ if $M \neq S + L$ for every proper submodule L of M. By Rad M we denote the sum of all small submodules of M or, equivalently the intersection of all maximal submodules of M. A ring R is said to be semilocal if R/ Rad R is semisimple. R is semilocal if and only if R has only finitely many maximal ideals. A module M is supplemented , if every submodule N of M has a supplement, i.e. a submodule K minimal with respect to $N + K = M$. K is a supplement of N in M if and only if $N + K = M$ and $N \cap K \ll K$. If $N + K = M$ and $N \cap K \ll M$ then K is called a weak supplement of N . M is a weakly essential supplemented module if every submodule of M has a weak essential supplement. By K we denote the set of all maximal ideals of R. Let R be a domain and M be an R-module. The submodule $T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$ is called the torsion submodule of M, and if M=T(M) then M is called a torsion module.

In this paper we show that the class of weakly essential supplemented modules need not be closed under extensions, that is if U and M/U are weakly supplemented for some submodule U of M then M need not be weakly essential supplemented. But if U has a weak essential supplement in M we show that M is weakly essential supplemented. We prove that a commutative ring R is semilocal if and only if every direct product of simple R-modules is weakly essential supplemented. Let R be a Dedekind domain. We obtain that an R-module M is weakly essential supplemented if and and only if T (M) and M/T (M) are weakly essential supplemented and T (M) has a weak essential supplement in M. If M is a torsion R-module with Rad $M \ll M$ then every submodule of M is weakly essential supplemented.

Extensions of weakly essential supplemented modules

A submodule N of a module M is called closed in M if $N \trianglelefteq K$ for some $K \subseteq M$ implies K=N. A submodule N of M is called coclosed in M if $N/K \ll M/K$ for some $K \subseteq M$ implies K = N.

Theorem 2.1: Let $0 \to L \to M \to N \to 0$ be a short exact sequence. If L and N are weakly essential supplemented and L has a weak essential supplement in M then M is weakly essential supplemented.

If L is coclosed in M then the converse holds, that is if M is weakly essential supplemented then L and N are weakly essential supplemented.

Proof. Without restriction of generality we will assume that $L \subseteq M$. Let S be a weak supplement of L in M i.e. $L + S = M$ and $L \cap S \ll M$. Then we have,

$$M/(L \cap S) = L/(L \cap S) \oplus S/(L \cap S)$$

$L/L \cap S$ is weakly essential supplemented as a factor module of L. On the other hand, $S/(L \cap S) \cong M/L \cong N$ is weakly essential supplemented. Then $M/(L \cap S)$ is weakly essential supplemented as a sum of weakly essential supplemented modules. Therefore M is weakly essential supplemented.

Suppose that L is coclosed. Then $L \cap S \ll L$ i.e. L is a supplement of S in M. Therefore L is weakly essential supplemented.

Proposition 2.2. Let R be a semilocal ring (not necessarily commutative) and M be an R-module. Suppose $U \subseteq M$ such that M/U is finitely generated. If U is weakly essential supplemented then M is weakly essential supplemented.

Proof. Suppose M/U is generated by

$$m_1 + U, m_2 + U, \ldots, m_n + U.$$

For the submodule $K = Rm_1 + Rm_2 + \ldots\ldots + Rm_n$ we have U + K = M. Then M is weakly essential supplemented.

The following well known lemma is given for completeness.

Lemma 2.3. Let M be a module and U be a finitely generated submodule of M contained in Rad M. Then U is small in M.

A module M is said to be locally noetherian if every finitely generated submodule of M is noetherian.

Proposition 2.4. Let M be a locally noetherian module and $X \subseteq$ Rad M. Suppose M/X is finitely generated. If X and M/X are weakly essential supplemented then M is weakly essential supplemented.

Proof. Since M/X finitely generated, X + L = M for some finitely generated submodule L of M. Then $X \cap L \subseteq X \subseteq$ Rad M is finitely generated, because L is finitely generated and M is locally noetherian. So $X \cap L \ll M$. Thus L is a weak essential supplement of X in M. Therefore M is weakly essential supplemented by Theorem 2.1.

We shall give an example in order to prove that the class of weakly essential supplemented modules need be closed under extensions. The following lemmas will be useful to present this example.

Lemma 2.5. Let R be a Dedekind domain. For an R- module M the following are equivalent:

1. M is injective,
2. M is divisible,
3. M = PM for every maximal ideal P of R,
4. M does not contain any maximal submodule.

Note that if M is divisible module over a Dedekind domain then Rad M =M. Hence if N is a module with Rad N= 0 then N does not contain divisible submodule.

Lemma 2.6. Let R be a domain and $\mathfrak{p}$ a maximal ideal of R. Then for every $\mathfrak{p}$ -primary R-module M, M/ Rad M is semisimple.

Corollary 2.7. Let R be a Dedekind domain and M a torsion R-module, then M/ Rad M is semisimple.

Proof. Since R is a Dedekind domain and M a torsion R-module, we have

$$M = \oplus_{\mathfrak{p}\epsilon\Omega} T_{\mathfrak{p}}(M)$$

Then

$$M/\,\mathrm{Rad}\, M = [\ \oplus_{\mathfrak{p}\epsilon\Omega} T_{\mathfrak{p}}(M)\ ] / [\ \oplus_{\mathfrak{p}\epsilon\Omega} \mathrm{Rad}\, T_{\mathfrak{p}}(M)] \cong \oplus_{\mathfrak{p}\epsilon\Omega}[T_{\mathfrak{p}}(M) \,/\, \mathrm{Rad}\, T_{\mathfrak{p}}(M)]$$

is semisimple by Lemma 2.6.

Lemma 2.8. Let R be a Dedekind domain and K be the field of quotients of R. Then $_RK$ is weakly essential supplemented.

Proof. Since R is a Dedekind domain and K/R is a torsion R-module, we have $K/R \cong \oplus_{\mathfrak{p}\epsilon\Omega} T_{\mathfrak{p}}(K/R)$ so K/R is supplemented by Theorem 2.4. Since R is finitely generated and Rad K = K we have $R \ll K$. Therefore K is weakly essential supplemented by Proposition 2.2 .

Lemma 2.9. Let R be a Dedekind domain and $\{\mathfrak{p}_i\}_{i\in I}$ be an infinite collection of distinct maximal ideals of R. Let $M = \prod_{i\in I} (R/\mathfrak{p}_i)$ be the direct product of the simple R-modules $R/\mathfrak{p}_i$ and T = T (M) be the torsion submodule of M. Then the following hold:

(1) M/T is divisible, therefore $M/T \cong K^J$ for some index set J.

(2) Rad M =0.

Proof. (1) Let be a maximal ideal of R. Then Then $\mathfrak{p}(M/T) = (\mathfrak{p}M + T)/T$. Now if $\mathfrak{p}$ is not one of the ideals $\{\mathfrak{p}_i\}_{i\in I}$ then $\mathfrak{p}M + T = M$ and so $\mathfrak{p}(M/T) = M/T$. Suppose $\mathfrak{p} \in \{\mathfrak{p}_i\}_{i\in I}$ say $\mathfrak{p} = \mathfrak{p}_j$ for some $j \in I$ then $\mathfrak{p}M = M(\bar{j})$ where $M(\bar{j})$ consists of of those elements of M whose j th coordinate is zero. Let M(j) be the submodule of M whose all coordinates except j th are zero. Clearly $M(j) \subseteq T$. Then $M = M(\bar{j}) + M(j) \subseteq \mathfrak{p}M + T$, so $\mathfrak{p}M + T = M$ and hence $\mathfrak{p}(M/T) = M/T$. Therefore by Lemma 2.5 M/T is divisible, and since it is torsion-free we have $M/T \cong K^J$.

(2) $M/M(\bar{j}) \cong R/\mathfrak{p}_j$ is a simple module, so $M(\bar{j})$ is a maximal submodule of M for every $j \in I$. Then we get $\mathrm{Rad}\, M \subseteq \cap_{j\in I} M(\bar{j}) = 0$.

Lemma 2.13. Let R be a domain and M be an R-module. Then the torsion submodule T (M) of M is closed in M.

Note that over a Dedekind domain a submodule is closed if and only if it is co-closed.

Proposition 2.14. Let R be a Dedekind domain and M be an R-module.Then the following holds.

1. If M is weakly essential supplemented then T (M) and M/T (M) are weakly essential supplemented. If T (M) has a weak essential supplement in M then the converse holds.
2. If Rad T (M)≪ M then M is weakly essential supplemented if and only if T(M) has a weak essential supplement in M and M/T (M) is weakly essential supplemented.
3. Suppose M is torsion. Then M is weakly essential supplemented if Rad M is weakly essential supplemented and has a weak essential supplement in M.
4. Suppose M/ Rad M is finitely generated and Rad M ⊴ M. Then M is weakly essential supplemented if Rad M is weakly essential supplemented.

Proof. (1) Suppose M is weakly supplemented. Then T (M) is a weak supplement in M. Since T (M) is also coclosed it is a supplement in M by ([5], Lemma 1.1). Then T (M) and M/T (M) are weakly supplemented by Proposition 2.2(5) in [8].

(2) If T (M) has a weak supplement then M is weakly supplemented by Theorem 2.1.

T (M)/ Rad T (M) is semisimple by Lemma 2.7 so it is weakly supplemented. Then T (M) is weakly essential supplemented by Proposition 2.2(4) in [8]. Then the proof is clear by (1).

(3) By Lemma 2.7 M/ Rad M is semisimple. Then the proof is clear by Theorem 2.1.

(4) Suppose M/ Rad M is generated by

$$m_1 + \text{Rad } M,\ m_2 + \text{Rad } M, \ldots, m_n + \text{Rad } M$$

Then for the finitely generated submodule $K = Rm_1 + Rm_2 + Rm_n$ we have Rad M + K = M and K ∩ Rad M is finitely generated as K is finitely generated, so K ∩ Rad M≪ M by Lemma 2.3 i.e. K is a weak essential supplement of Rad M in M.

By ([2] Proposition 9.15) Rad(M/ Rad M)=0, and since Rad M ⊴ M, M/ Rad M is torsion. Therefore M/ Rad M is semisimple by Lemma 2.7. Hence M is weakly essential supplemented by Theorem 2.1.

A module M is called coatomic if every proper submodule of M is contained in a maximal submodule of M. Over a commutative noetherian ring every submodule of a coatomic module is coatomic (see Lemma 1.1 in [15]). Note that coatomic modules have small radicals.

Proposition 2.15. Let R be a Dedekind domain and M be a torsion R- module. If Rad M ≪ M then every submodule of M is weakly essential supplemented.

Proof. The module M/ Rad M is semisimple by Lemma 2.7. Since Rad M ≪ M, every submodule of M is contained in a maximal submodule i.e. M is coatomic. Let N be a submodule of M. Then N is coatomic so Rad N≪ N , and since N is torsion, N/ Rad N is semisimple. Hence N is weakly essential supplemented by Proposition 2.2(4) in [8].

A domain R is said to be one-dimensional if R/I is artinian for every nonzero ideal I of R. One-dimensional domains are proper generalizations of Dedekind domains.

Lemma 2.16. Let R be a ring,I ≪ R and M be an R-module. If IM has a weak supplement K in M, then K is a weak essential supplement of I^nM in M for every $n \leq 1$.

Proof. By hypothesis IM + K = M and $I^2M + IK = IM$,so $I^2M + IK + K = IM + K$ which gives $I^2M + K = M$. Continuing this way we get:

$$I^nM + K = M \text{ and } I^nM \cap K \subseteq IM \cap K M.$$

This means that K is a weak essential supplement of I^nM in M.

Proposition 2.17. Let R be a one-dimensional domain and M be an R-module. Suppose that I is a nonzero ideal of R. If I^nM is weakly essential supplemented and I^kM has a weak essential supplement in M for some $k \leq n$, then M is weakly essential supplemented.

Proof. Since R is a domain $I^n \neq 0$. So R/I^n is an artinian ring because R is one-dimensional. Then M/I^nM is a supplemented R/I^n-module by Theorem 24.25 in [7] and Theorem 4.41 in [9]. Hence M/I^nM is a weakly essential supplemented R-module. By Lemma 2.16, I^nM has a weak essential supplement in M. Therefore by Theorem 2.1, M is weakly essential supplemented.

Corollary 2.18. Let R be a one-dimensional domain and M be an R- module. If rM is weakly essential supplemented for some 0≠ r ∈ R and has a weak supplement in M then M is weakly essential supplemented.

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