Study of Convex hull of a Set & Balanced set

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Abstract—*we define convex hull of a set and balanced set and established related important theorem. In this work we confine ourselves to that branch of functional Analysis which deals with some advancement in the theory of linear spaces and allied topics in functional Analysis. The whole work is based on the following new types of set called C-Set. C-Set in a real or complex linear space. It is the generalisation of convex set.*

I. INTRODUCTION

This research work concerns that branch of functional analysis, which deals with the theory of linear spaces. The whole work has been divided into four parts. In this paper we have introduced a new type of set, called C-Set in a real or complex space. We study C-Set in linear and Hilbert spaces.

If A be a Non-empty sub set of a linear space L. If for $x, y \in A,\ \alpha x - \beta y \in A,$ where $0 \le \alpha$ and $0 \le \beta$ and $\alpha + \beta = 1$, then we define A to be a C-Set. Choosing $\alpha = \beta = \frac{1}{2},\ y = x$, then $\frac{1}{2}x - \frac{1}{2}x = 0 \in A.$.

Choosing $\alpha = 0$, $\beta = 1$, we see that $-y \in A.$ also, taking –y in place of y, $\alpha x - \beta(-y) = \alpha x + \beta y \in A.$ Hence a C-Set is also a convex set. But as it is clear from the definition, a convex set is not necessarily a C-Set. Thus a C-Set is more general than a convex set.

Also 0 is an element of every C-Set which is not true in case of a convex set. It is also clear that a non-empty set A is a C-Set if and only if for $\alpha \ge 0,\ \beta \ge 0,\ \alpha + \beta = 1,\ \alpha x \pm \beta y \in A.$ We have defined the hull of a set A, denoted by $C_H(A)$, in a linear space L. $C_H(A)$ is the set of all linear combination of members of A, that is, the set of all sums $\alpha_1 x_1 \pm \alpha_2 x_2 \pm \ldots\ldots \pm \alpha_n x_n$ in which $x_i \in A, \alpha_i \ge 0$ and $\sum_{i=1}^{n} \alpha_i = 1$ n being arbitrary.

A few theorems concerning hull are as follows: If A be set in linear space L. Then $C_H(A)$ is a C-Set. Let A be a set in a linear space L. Then $C_H(A)$ is the intersection of all C-Sets containing A. If A and B are subsets of a linear space L such that

$A \subseteq B$ then $C_H(A) \subseteq C_H(B)$.

Definition: (I).The Convex hull of a set A in a Vector L in sort denoted by C(A) is the set of all sums .

$t_1x_1 + t_2x_2 + \ldots\ldots + t_nx_n$, in which $x_i \in A$, $t_i \geq 0, \sum t_i = 1; n$ being arbitrary.

(II). Let A be a Sub set of a linear space L then the Smallest balanced set containing A is called the balanced extension of A, in sort is denoted by B (A).

Theorem: Let A be a subset of a linear space L,

Then $B(C_H(A)) \subseteq (C_H B(A))$

Proof: First we show that $B(A) = \bigcup_\alpha \{\alpha A : |\alpha| \leq 1\}$ Let $D = \bigcup_\alpha \{\alpha A : |\alpha| \leq 1\}$ Taking $\alpha = 1, A \subseteq D$.Let $x \in D$ then $x = \alpha a$ for $a \in A$ and some α such that $|\alpha| \leq 1$.Let β be a scalar such that $|\beta| \leq 1$. Then $\beta x = \beta\alpha A$, where $|\beta\alpha| = |\beta|.|\alpha| \leq 1$.

Therefore $\beta x \in \beta\,\alpha\,A \subseteq D$.Hence D is balanced. Let S be a set such that $A \subseteq S$ and S is balanced. Let $x \in D$ then $x = \alpha a$, where $a \in A, |\alpha| \leq 1$. But $A \subseteq S \Rightarrow a \in S$ Since S is balanced, $\alpha a \in S$, Thus $x \in S$.Hence $x \in D \Rightarrow x \in S$. Therefore $D \subseteq S$.Hence D is the smallest balanced set containing A. So we conclude that

$$B(A) = D = \bigcup_\alpha \{\alpha A : |\alpha| \leq 1\}$$

Let z be an element of $B(C_H(A))$.From above result $B(C_H(A)) = \bigcup_\alpha \{\alpha C_H(A) : |\alpha| \leq 1\}$.So we see that $z \in \alpha C_H(A)$ for some α such that $|\alpha| \leq 1$. Hence we can write

$z = \alpha t_1 a_1 \pm \alpha t_2 a_2 \pm \ldots\ldots\ldots \pm \alpha t_n a_n$, where $a_i \in A, t_i \geq 0$ and $\Sigma t_i = 1$

Hence $z = t_1(\alpha a_1) \pm t_2(\alpha a_2) \pm \ldots\ldots\ldots\ldots \pm t_n(\alpha a_n)$. Now $a_i \in A \Rightarrow \alpha a_i \in B(A)$.

Therefore $z \in C_H(B(A))$. Thus $z \in B(C_H(A)) \Rightarrow z \in C_H(B(A))$.

Hence $B(C_H(A)) \subseteq C_H(B(A))$.