A NEW METHOD OF SEPARATION OF VARIABLES FOR THE DETERMINATION OF WAVE MOTION IN PERIODIC STRUCTURES

John Daniel, D.Sc. degree researcher, Mumbai University, Mumbai, India.

Abstract: Perturbation methods were used to study the sound wave propagation in several acoustic wave guides with weak surface undulations. These methods are highly complicated and applicable only in a very narrow domain of the wave guide. Moreover, such solutions create singularities in the solution. Singularities in solution of physical problems originate due to unrealistic assumptions. Therefore, a much simpler and widely applicable analytical method of solution was developed using separation of variables method and presented recently to such wave guide problems. Then application of that solution to find wave propagation in several periodic and non periodic waveguide structures is enumerated. In this article, a new and simple method of solution based on separation of variables and it's applications to analyze wave propagation problems in several waveguides with periodically variable boundaries and medium are explained.

Index terms: **A new method of separation of variables and wave motion in periodic structures.**

I. INTRODUCTION

A. H. Nayfeh, O. A. Kandil, A. M. Nusayr, G.V. Anand, M. K. George and V. Sundaravadivel [1 2 3 4 5 6 7] had studied the propagation of sound waves in simple acoustic wave guide with weak waves at the boundary using analytical methods. They used perturbation methods which are applicable only to wave guides with low frequency and smaller amplitude surface waves. Moreover, perturbation method of solution generates singularity in the solution and complicates the determination of solution at points close to the singularity. Therefore, later on John Daniel [8] developed a simpler analytical solution based on the method of separation of variables. Then application of that solution to find wave propagation in several periodic waveguide structures is enumerated [9]. In this article another new method of solution based on separation of variables is presented and it's application to solve several wave propagation problems is presented. First the theory based on the method of separation of variables to solve the wave propagation problem in a ocean acoustic waveguide with a wavy surface is presented and then the application of the method to solve wave propagation problems in several periodic acoustic, radio and optical waveguides is explained.

II. THEORY

Consider an oceanic wave guide with a wavy surface as shown in the following Figure-1. The surface wave is a single frequency wave propagating in x direction which can be expressed mathematically as

$$Z = h(x, t) = h_0 + a \cos(\alpha x - \Omega t) \tag{1}$$

Where h_0 is average channel depth, α is wave number, Ω is circular frequency and a is amplitude of the surface wave. Let ρi, Ci (i = 1,2) denote respectively the density and velocity of sound in the two media. Medium 2 is assumed to be a semi-infinite one. The interface between medium 1 and medium 2 is assumed to be flat.

Let us assume that a plane sound wave propagates in the wave guide in the direction x. The acoustic pressure P (x, y, z, t) can be determined by solving the wave equation.

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} - \frac{1}{C^2}\frac{\partial^2 P}{\partial t^2} = 0 \tag{2}$$

with boundary conditions

$$P(x, y, z, t) = 0 \text{ at } z = h(x, t) \tag{3.1}$$

$$P(x, y, +0, t) = P(x, y, -0, t) \tag{3.2}$$

$$V_z\Big|_{z=+0} = V_z\Big|_{z=-0} = \frac{1}{\rho_1}\frac{\partial P}{\partial z}\Big|_{z=+0} = \frac{1}{\rho_2}\frac{\partial P}{\partial z}\Big|_{z=-0} \tag{3.3}$$

where Vz is z component of particle velocity

$$P \rightarrow 0 \text{ a } z \rightarrow -\infty \tag{3.4}$$

Where +0 and -0 indicates that the interface z = 0 is approached from the sides z > 0 and z < 0 respectively.

Since there is no variation of pressure in y direction, equation (2) can be written as

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} - \frac{1}{C^2}\frac{\partial^2 P}{\partial t^2} = 0$$

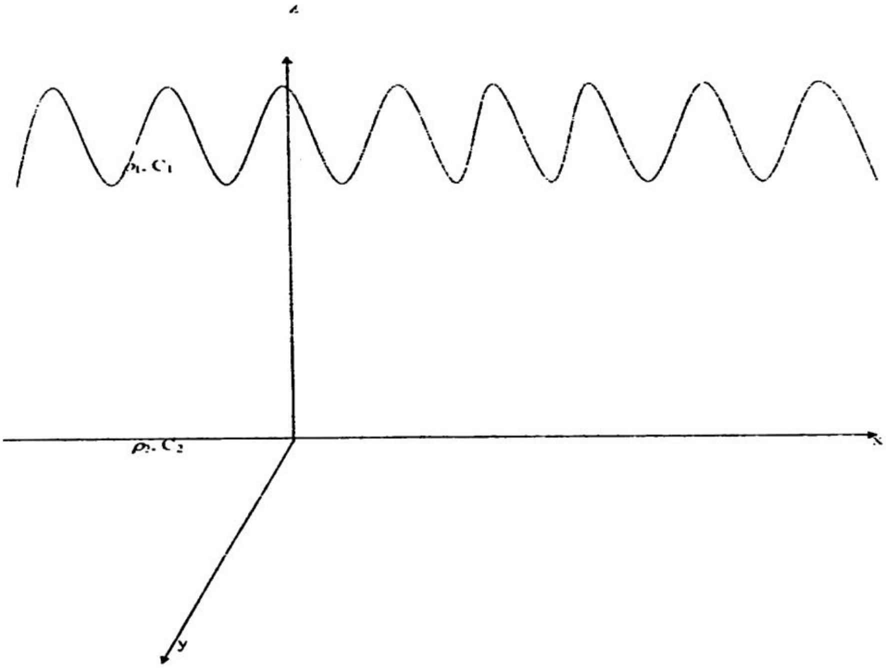

Figure-1

Therefore, equation (4) can be written as

$\partial^2P/\partial x^2 + \partial^2P/\partial y^2 + k^2P = 0$

Where (5)

$$k = \frac{\omega}{c_1} = k_1 \;\; for\; 0 < z < h \tag{6.1}$$

$$= \frac{\omega}{c_2} = k_2 \;\; for \;\; z < 0 \tag{6.2}$$

Therefore, by separation of variables methods,

$P\ (x, z, t) = \Psi_n\ (z).P_x(\ x)$ (7.1)

$P_x(x) = Sin(\zeta_n.x).\ Cos(n.\alpha.x - \Omega t)$ (7.2)

The function $\Psi_n(z)$ must satisfy the equation

$\partial^2\psi_n/\partial z^2 + (k^2 - \zeta_n{}^2 - (n\alpha)^2).\psi_n = 0$ (8)

with the boundary conditions

$\psi_n = 0$ at $z = h_0$ (9.1)

$\psi_n\ (+0) = \psi_n\ (-0)$ (9.2)

$(1/\rho_1).\partial\psi_n/\partial z = (1/\rho_2).\partial\psi_n/\partial z$ (9.3)

$\Psi_n \rightarrow 0$ as $z \rightarrow -\infty$ (9.4)

The solution to the equation (8) is

$\Psi_n = N_n \sin \chi_n\ (z/h_0 - 1)$ for $0 < z < h_0$

$= Cn\ e^{Dnz}$ for $z < 0$ (10)

Where N_n, C_n, χ_n & D_n, are constants and Ψ_n must satisfy the ortho normality condition.

$$\int_{-\infty}^{h} \rho\,(z)^{-1}\psi_n\ \psi_m\ dz = \delta_{mn} \tag{11}$$

Where $\rho(z)$ $= \rho_1$ for $0 < z < h$

$= \rho_2$ for $z < 0$

and δ_{mn} is kronecker delta.

By substituting the equating (10) into equation (1) and (8) we get

$$\int_0^h N_n^2\ \rho_1^{-1} \sin^2 \chi_n (\frac{z}{h} - 1)\ dz + \int_{-\infty}^{h} \rho_2{}^{-1}\ C_n^2\ e^{2Dnz}\ dz = 1 \tag{12}$$

$\zeta_n{}^2 = k_1{}^2 - (\alpha n)^2 - (\chi_n/h_0)^2$ (13)

$D_n = (\zeta_n{}^2 + (\alpha n)^2 - k_2{}^2)^{1/2}$ (14)

Substitution of equation (10) into equations (9b) & (9c) gives,

$-N_n \sin\chi_n = C_n$ (15)

$$\frac{1}{\rho_1} N_n \left(\frac{\chi_n}{h}\right) Cos\chi_n = C_n \frac{D_2}{\rho_2} \qquad (16)$$

Equations (13) to (16) can be combined to get

Cot χ_n = - (q χ_n)$^{-1}$ $(h^2(K_1^2 - K_2^2) - \chi_n^2)^{1/2}$ (17)

where q = ρ_2 / ρ_1 From equation (12) we get

$$N_n = \sqrt{\frac{2}{h\rho_1^{-1}(1 - \frac{\sin 2\chi_n}{2\chi_n}) + \frac{\rho_1^{-1}\sin^2\chi_n}{D_n}}} \qquad (18)$$

χ_n can be found by solving the equation (17).

Thus the complete analytical solution to the problem finding wave propagation in the waveguide is obtained. For $C_n = 0$, $\Psi_n = 0$ at z = 0. Therefore, sin $\chi_n = n\pi$, where n is a positive integer.

The solution is P (x, z, t) = Sin χ_n (z/h_0 – 1). Cos((α.n.x) - Ωt). Sin(ζ_n.x - ωt) (19)

III. APPLICATION-I

A. H. NAYFEH [1 2] DEVELOPED perturbation method to determine acoustic field inside a two dimensional acoustic waveguide with weakly undulating hard walls. This method is valid only for a waveguide with smaller waveguide undulations and the direct perturbation expansion results in solution which contains singularities at certain frequencies. At these resonant frequencies, method of multiple scales is used to find solutions at frequencies close to singularities. Such solutions are highly inaccurate at frequencies close to the singularities. As per the theory presented in this paper, solution to such wave motion problems is

P(x, z, t) = sin((z/h_0-1).γ_z).cos(n.α.x). cos(n.β.x +θ).sin(γ_xx – ωt),where $\gamma_z = n\pi$, $\gamma_x = ((\gamma_z)^2 - (n\alpha)^2 - (n\beta)^2 - (\omega/c)^2)^{1/2}$ (20)

and boundaries are assumed to be at z = h_1(x) = a.cos(αx) and z = h_2(x) = h_0+b.cos(βx+θ), where a, b, θ, α, β and h_0 are constants (21)

A.H.Nayfeh assumed that h_1(x) = a.cos(αx) and h_2(x) = a.cos(αx+θ) (22)

Therefore, solution presented here is more generalized solution.

IV. APPLICATION-II

A.H.Nayfeh and O.A.Kandil [3] developed perturbation method to find the acoustic field inside a circular cylindrical waveguide with weakly undulating waveguide walls. As per theory presented in this paper the generalized solution to such wave propagation problems is

P(r, φ, z, t) = J_n(r.h).($A_n$cos(n.φ) + B_n.sin(n.φ)).sin(γ_z.z – ωt). sin(n.k_1.z) where J_n(a.h) = 0, r_0(x) = a + b.Σ.α_m.sin(k_mx), $\gamma_z = (k^2 - (nk_1)^2 - h^2)^{1/2}$, k = ω/c, m is an integer, α_m, k_m, a and b are constants and r_0(x) is a periodic function (23)

J_n(r.h) is a Bessel's function of first kind of order n.

V. APPLICATION-III

A. M. Nusayr [4] applied perturbation methods developed by A. H. Nayfeh to solve wave propagation problem in a rectangular waveguide with weak hard wall undulations. To this wave propagation problem generalized solution is

P(x,y,z,t)=sin((z/h_0-1).γ_z).sin((y/w_0-1).γ_y).cos(n.α.x). cos(β.n.x+θ). cos(m.α_1.x). cos(m.β_1.x+θ_1).(sin(γ_xx – ωt), where $\gamma_z = n\pi$, $\gamma_y = m\pi$, $\gamma_x = ((\gamma_z)^2 + (\gamma_y)^2 - (n\alpha)^2 - (n\beta)^2 - (m\alpha_1)^2 - (m\beta_1)^2 - (\omega/c)^2)^{1/2}$ (24)

and boundaries are assumed to be at z = h_1(x) = a.cos(αx) and z = h_2(x) = h_0+b.cos(βx+θ), y = h_3(x) = c.cos(α_1.x) and y = h_4(x) = w_0+d.cos(β_1.x+θ_1), where a, b, c, d, θ, θ_1, α, α_1, β, β_1, h_0 and w_0 are constants (25)

Waveguide dimensions h_1, h_2, h_3, and h_4 are periodic functions of x. Therefore, solution presented here is much more generalized solution as compared with that of A.M.Nusayr.

VI. APPLICATION-IV

M.A. Hawwa [5] has applied perturbation methods developed by Nayfeh, et al to study the wave propagation with a periodic square wall profile. Since the method developed in this article is applicable to waveguide with any type of periodic boundary variations, solutions developed in sections applications-I, II, III are valid to two dimensional, rectangular and circular cylindrical waveguides with any type of periodic boundary variations.

VII. APPLICATION-V[6 7 8 9]

In the theory presented in the first section, an underwater acoustic waveguide with a travelling sinusoidal surface on the top boundary of the waveguide and a semi infinite medium from the bottom flat boundary were assumed. However, in practice bottom floor of the ocean can't be flat. Therefore, assumption of variable boundary at the oceanic floor will be closer to natural situation. Since the method developed in this article is applicable to waveguides with any type of periodic boundary variations, the generalized solution is applicable to ocean acoustic waveguides with any type of periodic surface and ocean floor variations.

VIII. APPLICATION-VI[9 10 11]

One dimensional Photonic Band Gap (P.B.G.) materials with periodic defects and bound by metal planes on both sides are analyzed for their filtering behavior to develop nano scale electronic filters. Since dielectric medium varies periodically in the direction of propagation of electromagnetic wave, the velocity of electromagnetic signal could be described in the Fourier series form and the method of separation of variables could be applied to determine the solution to the wave propagation problem. The solution is E(x, z, t) = sin(γ_z.z).sin(n.(2π/l).x).sin(γ_xx – ωt), where $\gamma_z = n\pi/h_0$, $\gamma_x = ((n\pi/h_0)^2 - (2n\pi/l)^2 - k^2)^{1/2}$, k = ω/c, l is the period of dielectric variation in x direction and c = 3x108 m/s (26)

This solution is applicable to any type of periodic dielectric medium $\varepsilon(x)$ variations. The solution clearly indicates the dependency of filtering characteristics on dielectric medium variations. Therefore, by proper choice of c(x) all types of filters could be constructed.

IX. APPLICATION-VII [9 10 11 12]

One and two dimensional PBG materials with air-dielectric interface and periodic defects are analyzed for their filtering behavior. The solution for TM wave propagation in one dimensional PBG material bound by air-dielectric interface is $H(x, z, t) = \sin(\gamma_z.z).\ \sin(\gamma_x x - \omega t).\sin((2n\pi/l).x)$ or $\cos(\gamma_z.z).\sin((2n\pi/l).x).\sin(\gamma_x.x - \omega t)$ for $0 < z < h_0$ and $H(x, z, t) = H_0.e^{-\gamma z}$ for $z > h_0$ and $= H_0 e^{\gamma z}$ for $z < 0$ where h_0 is the height of the dielectric material of the 1-D PBG material and γ_x, γ_z, γ and H_0 are constants

(27)

The unknown constants could be found by substituting the solution at the boundaries and in the wave equation. In the wave equation, $k = \omega/c$ and $c = 3\text{x}10^8$ m/s. This solution is applicable to any type of periodic dielectric medium $\varepsilon(x)$ variations. The solution clearly indicates the dependency of filtering characteristics on dielectric medium variations. Therefore, by proper choice of c(x) all types of filters could be constructed.

X. CONCLUSION

A much simpler analytical solution to the problem of finding wave propagation in several acoustic, radio and optical waveguides with periodic structures is explained which is applicable to periodic structures of any dimensions and parameter values without any singularity in the solution. The method of solution could be extended easily to 2 and 3 dimensions also. The method could be extended to analyze the ionosphere radio wave propagation and also to analyze the light wave propagation in optical waveguides with surface irregularities.

XI. ACKNOWLEDGEMENT

The author is grateful to Prof. G. V. Anand, Indian Institute of Science, Bangalore, India and Prof. R. H. Macphie, University of Waterloo, Canada for introducing ocean acoustic waveguide problems and Waveguide discontinuity scattering problems, Indian Institute of Science, Bangalore and University of Waterloo, Canada for a research scholarship/assistantship during 1987-1990.

REFERENCES

[1]. A.H.Nayfeh.1973.Perturbation Methods. Wiley Inter-science, Chapter-6.
[2]. A.H.Nayfeh.1974.Sound waves in two dimensional ducts with sinusoidal walls. J.A.S.A., 56(3), September, pp.768-770.
[3]. A.H.Nayfeh, O.A.Kandil 1978. Propagation of waves in cylindrical hard walled ducts with generally weak undulations. AIAA Journal, Vol.16, No.10, October, pp.1041-1045.
[4]. A.M.Nusayr.1980. Propagation of waves in rectangular ducts with sinusoidal undulations. J.A.S.A., Vol.67, No.5, 1980, pp. 1472-1476.
[5]. M.A. Hawwa.2015. Sound propagation in a duct with wall corrugations having square wall profiles. Hindawii publishing corporation, Mathematical problems in Engineering, Volume-2015, Article Id: 516982, 8 pages.
[6].G.V. Anand and M.K. George.1986. Normal mode sound propagation in an ocean with sinusoidal surface waves. J.A.S.A. 80 (Part –1), 238 – 243.
[7].V. Sundaravadivel.1990. Normal mode wave propagation in an ocean with a wavy surface. M.Sc (Engg.) thesis, E.C.E. Dept., I.I.Sc., Bangalore, India.
[8]. John Daniel.2018. Frequency Spectrum of normal mode sound propagation in an ocean with a wavy surface. Submitted to Applied Acoustics, Elsevier publishers, Netherlands.
[9]. John Daniel.2018. Application of method of separation of variables to find wave motion in waveguides with variable boundaries and medium. Submitted to Wave Motion, Elsevier publications, Netherlands.
[10]. K. Srivatsava.2007. Omnidirectional Reflection Band in 1-Dimensional Photonic Crystal Structures using Fullerene Films. Progress in Electromagnetic Research, PIER-74, pp. 181-194.
[11]. Fabrizio Frezza, et al.2005. Numerical Investigation on the Filtering behaviour of 2-D PBG with multiple periodic defects., IEEE transactions on Nano technology, Vol.4, No.6, Nov.730-738.
[12]. E.C.Jordan and K.G.Balmain.2007.Electromagnetic Waves and Radiating Systems. Prentice Hall of India, New Delhi, pp. 273-275.