TIME FRACTIONAL BIOLOGICAL POPULATION MODEL BASED ON HPM: AN ANALYTICAL SOLUTION

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ABSTRACT: In this paper, we execute a relatively analytical technique, the homotopy perturbation method (HPM), for illuminating fractional Biological population model. The fractional derivatives are described in Caputo derivatives. The outcome reveals that the proposed method is very effective and simple for obtaining analytic solutions of nonlinear fractional partial differential equations. A few models are given to show capability of the method for solving the fractional nonlinear partial differential equation.

Keywords: Homotopy perturbation method (HPM), Caputo derivatives, Biological population Model, Mittag Leffler function.

1. INTRODUCTION

In the last couple of decades, fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearances in various applications in Fluid Mechanics, Viscoelasticity, Biology, Physics, Electrical network, Control theory of dynamical systems, Chemical physics, Optics and Signal processing, which can be successfully modelled by linear and non-linear fractional order differential equations. The fractional calculus has a tremendous use in basic sciences and engineering that has played a key role in development of the subject (Oldham et al, 1974). A few essential works to solving fractional differential equations have been under taken (Miller K. S.et al, 2003).

Our model leads to degenerate parabolic nonlinear partial differential equations arising in the spatial diffusion of biological populations as

$$D_t u = G(u)_{xx} + G(u)_{yy} + f(t,x,y,u),$$
$$0 < \alpha \le 1,\ \ t \ge 0,\ \ x, y \in R, \tag{1}$$

with initial condition $u(x, y,0) = u_0(x, y,0).$ where u denotes the population density and f represents the population supply due to birth and death. The movements are made generally either by mature animals driven out by invaders or by young animals just reaching maturity moving out of their parental territory to establish breeding territory of their own.

In both cases, it is much more plausible to suppose that they will be directed towards nearby vacant territory. In this model, therefore, movement will take place almost exclusively "down" the population density gradient, and will be much more rapid at high population densities than at low ones. In an attempt to model this situation, they considered a walk through a rectangular grid, in which at each step an animal may either stay at its present location or may move in the direction of lowest population density.

In this article, we will do it in a practical case $G(u)=u^2$ (Lu Yun Guang,2000), besides the theory of the spread of biological populations, the case $G(u)=u^2$ occurs in a variety of different setting. The function $G(u)$, has the property that $G'(0)=0, G'(p)>0,$ for $u>0,$ so that (1) is degenerate: it is a second order parabolic equations, where $u>0.$ But degenerates to first order when $u=0.$ We consider a more general form of $f(u)=hu^g(1-ru^h).$ After this assumption, equation (1) will be

$$D_t u=(u^2)_{xx}+(u^2)_{yy}+hu^g(1-ru^h), \quad 0<\alpha\le 1,\ t\ge 0,\ x,y\in R, \qquad (2)$$

with initial condition $u(x,y,0)=u_0(x,y,0).$

The objective of this paper is to extend the application of the homotopy perturbation method (HPM) to obtain analytic solution of the time fractional biological problem. The homotopy perturbation method was first proposed by the Chinese mathematician He (He J. H., 1999) and was successfully applied to solve nonlinear wave equations by He (He J. H., 2005). Recently, many researchers applied numerical methods to find the numerical solution of biological problems such as numerical solution of a biological population model (F. Shakeri et al.,2007). Exact solution of fractional order biological population model (El- Sayed A.M.A. et al.,2009), Holder Estimates of Solutions of Biological population equations (Lu Yun Guang, 2000). On the Diffusion of Biological Populations (Gurtin M. E.et al,1997), Application of Homotopy Perturbation Method to Biological Population Model (Roul P 2010). In the present paper,

we have to solve the nonlinear time-fractional biological problem, employed (Gurney W. S.C.et al, 1975), by homotopy perturbation method. This equation can be written in operator form as:

$$D_t^\alpha u=D_x^2u^2+D_y^2u^2+hu^g(1-ru^h), \quad 0<\alpha\le 1,\ t\ge 0,\ x,y\in R, \qquad (3)$$

with initial condition $u(x,y,0)=u_0(x,y,0),$ where $u,$ denotes the populations density and $f,$ represents the population supply due to births and deaths.

2. PRELIMINARIES AND NOTATIONS:

In this section, we give some definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. A real function $f(x),\ x>0,$ is said to be in the space $C_\mu,\ \mu\in R,$ if there exists a real number $p(>\mu),$ such that $f(x)=x^p f_1(x),$ where $f_1(x)\in C[0,\infty),$ and it is said to be in the space C_μ^m if and only if $f^{(m)}\in C_\mu, m\in N.$

Definition2.2. The Riemann–Liouville fractional integral operator (j^α) of order $\alpha\ge 0,$ of the function $f\in C_\mu,\ \mu\ge -1,$ is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, \; x > 0,$$

$$J^0 f(x) = f(x).$$

Properties of the operator (j^{α}), we mentation only the following. For $f \in C_{\mu}, \; \mu \geq -1, \; \alpha, \; \beta \geq 0$ and $\gamma \geq -1$:

(a) $J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x),$

(b) $\left(J^{\alpha} J^{\beta}\right) f(x) = \left(J^{\beta} J^{\alpha}\right) f(x),$

(c) $J^{\alpha} x^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} x^{\gamma+\alpha}$

The Riemann–Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^{α} proposed by Caputo in his work in the theory of viscoelasticity (Y.Luchko, 1998).

Definition2.3. The fractional derivatives (D^{α}) of $f(x)$ in the Caputo's sense is defined as

$$D^{\alpha} f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}} dt, \quad \alpha > 0, \; x > 0,$$

for $m-1 < \mathrm{Re}\,(\alpha) \leq m, \; m \in N, \;\; f \in C_{-1}^{m}.$

The following are two basic properties of the Caputo's fractional derivative:

Lemma 2.1. If $m-1 < \alpha \leq m, \; m \in N$ and $f \in C_{\mu}^{n}, \; \mu \geq -1,$ then

$$(D^{\alpha} J^{\alpha}) f(x) = f(x),$$

$$(J^{\alpha} D^{\alpha}) f(x) = f(x) - \sum_{i=0}^{m-1} f^{i}(0^{+}) \frac{x^{i}}{i!},$$

The Caputo fractional derivatives are considered here because it allows traditional initial conditions to be included in the formulation of the problem.

Definition2.4. The Mittag-Leffler function (F.Mainardi,1994) $E_{\alpha,\beta}(z)$ is a special complex function which depends on two complex parameter α and β. It may be defind by the following series when the real part of α is strictly positive

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0,$$

3. Numerical Examples

In this section, to demonstrate the effectiveness of the HPM algorithm, two examples of the fractional biological problem are studied.

Example 3.1: Let us consider the following time fractional biological population model

$$D_t^{\alpha} u = D_x^2 u^2 + D_y^2 u^2 - u\left(1 + \frac{8}{9} u\right), \quad 0 < \alpha \leq 1, \qquad (4)$$

subject to the initial condition $u(x, y, 0) = \exp\left(\frac{1}{3}(x+y)\right)$

By homotopy perturbation method, we consider Eq (4) has solution

$$u = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + ... \tag{5}$$

According to the homotopy perturbation method, we can construct the homotopy for Equation (4)

$$D_t^\alpha u = p\left[D_x^2 u^2 + D_x^2 u^2 - u\left(1 + \frac{8}{9} u\right)\right], \tag{6}$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial x^\alpha}$, $D_x^2 = \frac{\partial^2}{\partial x^2}$, and $D_y^2 = \frac{\partial^2}{\partial y^2}$.

Substituting (5) into (6) and equating the terms with identical powers of p,

we obtain the following set of linear partial differential equations

$$p^0: \ D_t^\alpha u_0 = 0,$$

$$p^1: \ D_t^\alpha u_1 = \left(D_x^2 + D_y^2 - \frac{8}{9}\right) u_0^2 - u_0,$$

$$p^2: \ D_t^\alpha u_2 = \left(D_x^2 + D_y^2 - \frac{8}{9}\right)(2u_0 u_1) - u_1,$$

$$p^3: \ D_t^\alpha u_3 = \left(D_x^2 + D_y^2 - \frac{8}{9}\right)\left(u_1^2 + 2u_0 u_2\right) - u_2, ...$$

solving the above equations, we obtain the following approximations

$$u_0(x, y, t) = u(x, y, 0) = \exp\left(\frac{1}{3}(x + y)\right),$$

$$u_1(x, y, t) = -\exp\left(\frac{1}{3}(x + y)\right)\frac{t^\alpha}{\Gamma(\alpha + 1)},$$

$$u_2(x, y, t) = \exp\left(\frac{1}{3}(x + y)\right)\frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, ...$$

$$u_n(x, y, t) = (-1)^n \exp\left(\frac{1}{3}(x + y)\right)\frac{t^{n\alpha}}{\Gamma(n\alpha + 1)},$$

According to the HPM, we can obtain the solution in a series form as follows

$$u(x, y, t) = \exp\left(\frac{1}{3}(x + y)\right)\sum_{k=0}^{\infty}\frac{(-t^\alpha)^k}{\Gamma(\alpha k + 1)},$$

$$u(x, y, t) = e^{\frac{1}{3}(x+y)} \ E_\alpha(-t^\alpha).$$

This is the exact solution of Eq. (4).

If $\alpha = 1$ then $u(x, y, t) = e^{\frac{1}{3}(x+y)-t}$,

which is the exact solution of the standard biological problem (Roul P.,2010)

Example 3.2: Now we consider the following time fractional biological population model

$$D_t^{\alpha} u = D_x^2 u^2 + D_y^2 u^2 + hu, \ 0 < \alpha \le 1, \tag{7}$$

subject to the initial condition $u(x, y, 0) = \sqrt{x\,y}$.

By HPM, we can construct the homotopy for equation (7) which satisfies

$$D_t^{\alpha} u = p[D_x^2 u^2 + D_y^2 u^2 + hu], \tag{8}$$

substituting (5) into (8) and equating the coefficient of corresponding power of p, we gets

$$u_0(x, y, t) = u(x, y, 0) = \sqrt{xy},$$

$$u_1(x, y, t) = h\sqrt{xy}\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2(x, y, t) = h^2\sqrt{xy}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \ldots . \qquad u_n(x, y, t) = h^n\sqrt{xy}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},$$

According to the HPM, we can obtain the solution in a series form as follows

$$u(x, y, t) = \sqrt{x\,y}\sum_{k=0}^{\infty}\frac{(ht^{\alpha})^k}{\Gamma(\alpha k+1)},$$

$$u(x, y, t) = \sqrt{xy}\ E_{\alpha}(ht^{\alpha})$$

which is the exact solution of Eq (7) for.

If $\alpha = 1$ then $u(x, y, t) = \sqrt{xy}\ e^{ht}$

This is the exact solution of the standard biological problem of (Roul P., 2010)

5. CONCLUSIONS

In this paper, the Homotopy perturbation method has been successfully applied to find the solution of fractional biological problems. Thus it is obviously observed that HPM is a very powerful and efficient technique in finding analytical solutions.

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