An algorithmic approach for defining Julia set structure using the complex Newton-Raphson method

[1]Asimina Krimpeni, [2]Agathoklis A. Krimpenis
[1]Senior Researcher, [2]Assistant Professor
General Department, National and Kapodistrian University of Athens, 34400 Psachna Eyvoias, Greece

Abstract : Fractals can derive from the application of the complex Newton-Raphson method for solving nonlinear equations. The iterative "nature" of the method yields two main sets: convergence and divergence sets. The chaotic Julia set consists of the topological boundary of these two sets. In the present work, it is proven that the dynamical system of the Julia set and its complex Newton-Raphson transformation is, in fact, a chaotic dynamical system. An iterative algorithmic approach that helps obtain and represent graphically the Julia set is then presented.

***Keywords* - Complex Newton-Rapshon, Julia set, fractal, chaotic dynamical system, code space, shift dynamical system.**

I. Introduction

Strange phenomena have often been observed in solving nonlinear equations, even when implementing one as simple as the Newton-Raphson method. This can produce a complicated behavior of numerical solutions. The beginning appeared in a difficult period of Gaston Julia's life, when he was recovering from a severe injury from his World War I service. At that time, he published an article that described the iteration of a rational function [15]. It was in this paper that Julia first introduced the modern idea of a "Julia set". At the same period, another mathematician, Pier Fatou, created the area of Mathematics which is called holomorphic dynamics and it deals with a global study of analytic functions iterability. Moreover, Fatou introduced and studied the Julia set [20]. The subject was lulled after a while and it was not until Benoit Mandelbrot began studying iterability in early 1970's that Julia sets re-emerged. After that, there was a boom of interest in Julia sets in correlation with fractals ([2], [4], [10], [16]) and consequently with chaos ([1], [3], [7]). The simultaneous advancement and rapid evolution of computers created new research interest that involved chaotic phenomena and fractals in Numerical Analysis ([8], [9]), and of course their implementations in programming ([5], [6], [11]).

The well-known Newton-Raphson (N-R) algorithm, for complex functions of one variable, for finding the roots of $f(z) = 0$ is given by

$$z_k = z_{k-1} - \frac{f(z_{k-1})}{f'(z_{k-1})},$$

where f' is the complex derivative of f , z_0 is the point of initialization, k = 1, 2… is the iteration index.

The study of the complex Newton-Raphson leads to unexpected subsets of the complex field, with properties of the so-called "fractals". It also leads to dynamical systems with chaotic behavior and other intriguing properties. These properties have been studied extensively by Steven Smale ([12], [13], [14]).

Complex Newton-Raphson is applied in solving with iterative algorithms non linear equations in $\mathbb{C}$. So let $f(z) = 0$ be a given polynomial equation and define S to be the set of the initial values $z_0 \in \mathbb{C}$ of the corresponding iterative Newton-Raphson type, for which the sequence $\{z_n\}$ converges to one of the roots of the equation. Let also A be the set of the initial values $z_0 \in \mathbb{C}$ for which the sequence $\{z_n\}$ diverges. Then the boundary of S and A is called the Julia Set and it is symbolized with J.

As a paradigm, consider the equation $f(z) = z^3 - 1 = 0$. The application of N-R method yields S, A and J sets as defined above. In this case, A is identical to J, which has a "strange" shape (in fact, it is a fractal). The N-R method, when it is used for solving this equation in the real field, generally diverges for large regions of the initial $x_0 \in \mathbb{R}$, and converges when x_0 is "close enough" to a root. The fact that all the complex roots exist (Fundamental Theorem of Algebra) does not, a priori, imply that the complex N-R converges for almost every $z_0 \in \mathbb{C}$. In other words, the existence of solutions does not guarantee the existence of a converging algorithm for finding them. It is, thus, interesting, not only that the complex N-R converges for almost every initial value $z_0 \in \mathbb{C}$, but also that the corresponding Julia set (in this case, the set of all $z_0 \in \mathbb{C}$ for which the method diverges), is, as is shown, a chaotic dynamical system.

II. Definitions and Theoretical Background

Definition 1.
Let (X, d) a complete metric space and $H(X)$ the set of all of nonempty compact subsets of X. We define as Hausdorff metric of A and B , when $A, B \in H(X)$, the relation $h(A, B) = d(A, B) \vee d(B, A)$, where $d(A, B) = \max\{d(x, B): x \in A\}$ and $d(x, B) = \min\{d(x, y): y \in B\}$. Thus, we consider the topological space $(H(X), h(d))$, which is the fundamental space of deterministic fractals.

Lemma 1.

Let $w: X \to X$ be a contraction mapping on the metric space (X, d) with contractivity factor s. Then $w: H(X) \to H(X)$ defined by $w(B) = \{w(x): x \in B\}$ for every $B \in H(X)$ is a contraction mapping on $(H(X), h(d))$ with contractivity factor s.

Proof:

Every contraction mapping on a metric space is continuous, therefore $w: X \to X$ is continuous. Furthermore it is easy to be proven that w maps $H(X)$ into itself.

Let $B, C \in H(X)$, then

$$d\big(w(B), w(C)\big) = \max\{\min\{d\big(w(x), w(y)\big): y \in C\}: x \in B\}$$
$$\leq \max\{\min\{s \cdot d(x, y): y \in C\}: x \in B\} = s \cdot d(B, C).$$

Similarly, $d\big(w(C), w(B)\big) \leq s \cdot d(C, B)$

Therefore, $h(w(B), w(C)) \leq s \cdot h(B, C)$

Lemma 2.

For all A_1, A_2, A_3, A_4 in $H(X)$ $h(A_1 \cup A_2, A_3 \cup A_4) \leq h(A_1, A_3) \vee h(A_2, A_4)$, where h is Hausdorff metric.

Lemma 3.

Let (X, d) a complete metric space, and also, let $\{w_n,\ n = 1,2, \ldots, N\}$ be contraction mappings on $(H(X), h(d))$. The contractivity factor for w_n be denoted by s_n for each n. We define $W: H(X) \to H(X)$ by $W(B) = \bigcup_{n=1}^{N} w_n(B)$, for each $B \in H(X)$. Then W is a contraction mapping with contractivity factor $s = \max\{s_n: n = 1,2, \ldots, N\}$

Proof:

Let assume that $N = 2$, and let $B, C \in H(X)$, then we have

$$h\big(W(B), W(C)\big) = h(w_1(B) \cup w_2(B), w_1(C) \cup w_2(C)$$
$$\leq h(w_1(B), w_1(C)) \vee h(w_2(B), w_2(C))$$
$$\leq s_1 \cdot h(B, C) \vee s_2 \cdot h(B, C) \leq s \cdot h(B, C)$$

And by induction the proof is completed.

Theorem 2.

Let $\{X; w_n,\ n = 1,2, \ldots, N\}$ be a hyperbolic iterated function (IFS) with contraction factor $s = max\{s_n, n = 1,2, \ldots, N\}$, where $w_n: X \to X$ is a contraction mapping with contraction factor s_n , where $(\boldsymbol{X}, \boldsymbol{d})$ is a complete metric space.

Then the transformation $W: H(X) \to H(X)$ defined by

$$W(B) = \bigcup_{n=1}^{N} w_n(B), \text{ for each } B \in H(X),$$

is a contraction mapping on the complete metric space $(H(X), h(d))$ with contraction factor s . That is

$$h\big(W(B), W(C)\big) \leq s \cdot h(B, C) \text{ for all } B, C \in H(X).$$

Its unique fixed point, $A \in H(X)$, obeys $A = W(A) = \bigcup_{n=1}^{N} w_n(A)$ *and is given by* $A = \lim_{n\to\infty} W^{\circ n}(B)$, for all $B \in H(X)$, where $W^{\circ n}(B) = W(W(\ldots W(B)\ldots))$.

This unique fixed point $A \in H(X)$ is called the attractor of the IFS and it is a fundamental concept in the study of fractals.

In parallel to the "geometrical" structure of fractals, we consider a fractal code space, in order to explore the relation between a "geometrical" fractal and its corresponding "algebraic-encoded" form. This will be used for studying the chaotic structure of Julia sets.

A well known such analogy is easily seen in the construction of the Shierpinski triangle in which the attractor is a zero measure "cloud" of the code space and is generated by an alphabet of cardinality 3.

III. Chaotic Dynamical Systems

Definition 2.

Let Σ be the code space on N symbols $\{0,1,2, \ldots, N - 1\}$, where for a given element $X \in \Sigma$, we write $x = x_1x_2x_3x_4 \ldots$

There are infinitely many ordered entries x_i (chosen among N symbols) for each $\boldsymbol{x}$.

On the code space the expression $d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{(N+1)^i}$ for all $x, y \in \Sigma$, which is easy to show that it is a metric.

So, (Σ, d) is a metric space.

Definition 3.

On the metric space (Σ, d), we define a transformation $T: \Sigma \to \Sigma$

by $T(x) = T(x_1x_2x_3x_4\ldots) = x_2x_3x_4x_5 \ldots$for every $x = x_1x_2x_3x_4\ldots \in \Sigma$, which is called Shift Operator.

Then $\{\Sigma; T\}$ is a *dynamical system.*

Definition 4.

A dynamical system is a transformation $f: X \to X$ on a metric space (X, d) which is denoted by $\{X; f\}$.

Definition 5.

The orbit of a point $x \in X$ is the sequence $\{f^{\circ n}(x)\}_{n=0}^{\infty}$.

Definition 6.

A periodic point of f is a point $x \in X$ such that

$$f^{\circ n}(x) = x \text{ , for some } n \in \{1,2,3,\dots\}.$$

If x is a periodic point of f, then an integer n such that $f^{\circ n}(x) = x$, $n \in \{1,2,3,\dots\}$ is called a period of x.

The least such integer is called the minimal period of the periodic point x.

The orbit of a periodic point of f is called a cycle of f.

The minimal period of a cycle is the number of distinct points it contains. A period of a cycle of f is a period of a point in the cycle.

Definition 7.
Let (X, d) be a metric space. A sequence $\{x_n\}_{n=0}^{\infty}$ of points of X is said to be dense in X if, for each point $a \in X$, there is a subsequence $\{x_{\sigma_n}\}_{n=0}^{\infty}$ that converges to $a \in X$. In particular, an orbit $\{x_n\}_{n=0}^{\infty}$ of a dynamical system $\{X; f\}$ is said to be dense in X if the sequence $\{x_{\sigma_n}\}_{n=0}^{\infty}$ is dense in X.

Let (X, d) be a metric space.

Definition 8.
A dynamical system $\{X; f\}$ is transitive if, whenever U and V are open subsets of the metric space (X, d), there exists a finite integer n such that $U \cap f^{\circ n}(V) \neq \emptyset$.

Definition 9.
The dynamical system $\{X; f\}$ is sensitive to initial conditions if there exists $\boldsymbol{\delta} > 0$ such that, for any $x \in X$ and any ball $B(x, \varepsilon)$ with radius $\varepsilon > 0$, there is $y \in B(x, \varepsilon)$ and an integer $n > 0$ such that $d(f^{\circ n}(x), f^{\circ n}(y)) > \delta$.

Definition 10.
A dynamical system $\{X; f\}$ is chaotic if
1) it is transitive
2) it is sensitive to initial conditions
3) the set of periodic orbits of f is dense in X.

Theorem 3.
Let Σ be the metric code space on N symbols (alphabet) with metric $d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{(N+1)^i}$ for all $x, y \in \Sigma$ and let T be the swift operator $T: \Sigma \to \Sigma$ defined by $T(\sigma) = T(\sigma_1\sigma_2\sigma_3\sigma_4 \dots) = \sigma_2\sigma_3\sigma_4\sigma_5 \dots$ for all $= \sigma_1\sigma_2\sigma_3\sigma_4 \dots \in \Sigma$.
Then the shift dynamical system $\{\Sigma; T\}$ is *chaotic*.

IV. Association between Code Space and Shift Dynamical Systems

Let $\{X; \cdot\, w_n, n = 1,2, \dots, N\}$ be a totally disconnected IFS with attractor.

Definition 11.
The associated shift transformation on A is the transformation $S: A \to A$ defined by $S(a) = w_n^{-1}(a)$, for $a \in w_n(A)$, where w_n is viewed as a transformation on attractor A.

Definition 12.
The dynamical system $\{A;\ S\}$ is called the shift dynamical system associated with the IFS.

Definition 13.
The code space associated with the IFS, (Σ, dc), is defined to be the code space on N symbols $\{1,2,3, \dots, N\}$, with the metric d_c given by:

$$d_c = \sum_{n=1}^{\infty} \frac{|\tau_n - \sigma_n|}{(N+1)^n}, \text{ for all } \tau, \sigma \in \Sigma.$$

This association between the code space and shift dynamical system is basic for the following theorem.

Lemma 4.

Let $\{X;\ w_n: n = 1,2, \dots, N\}$ be a hyperbolic IFS, where (X, d) is a complete metric space. Let $K \in H(X)$. Then there exists $\widetilde{K} \in H(X)$ such that $K \subset \widetilde{K}$ and $w_n: \widetilde{K} \to \widetilde{K}$ for $n = 1,2, \dots, N$. In other words $\{\widetilde{K};\ w_n: n = 1,2, \dots, N\}$ is a hyperbolic IFS, where the underlining space is *compact*.

Lemma 5.

Let $\{X;\ w_n: n = 1,2, \dots, N\}$ be a hyperbolic IFS of contractivity s, where (X, d) is a complete metric space. Let (Σ, dc) denote the code space associated with the IFS. For each $\in \Sigma$, $n \in \mathbb{N}$ and $x \in X$ let $\Phi(\sigma,n,x) = w_{\sigma_1} \circ w_{\sigma_2} \circ w_{\sigma_3} \dots \circ w_{\sigma_n}(x)$. As K is denoted a compact non-empty subset of X. Then there is a real constant D such that $d\big(\phi(\sigma,m,x_1),\phi(\sigma,n,x_2)\big) \le Ds^{m\wedge n}$ for all $\in \Sigma$, all $m,n \in \mathbb{N}$, and all $x_1,x_2 \in K$.

Theorem 4.

Let (X,d) be a complete metric space. Let $\{X; w_n: n = 1,2,\dots,N\}$ be an IFS. Let, also, (Σ, dc) denote the code space associated with the IFS. For each $\in \Sigma$, $n \in \mathbb{N}$ and $x \in X$ let $\Phi(\sigma,n,x) = w_{\sigma_1} \circ w_{\sigma_2} \circ w_{\sigma_3} \dots \circ w_{\sigma_n}(x)$.
Then $\Phi(\sigma) = \lim_{n\to\infty} \Phi(\sigma,n,x)$

i) exists
ii) belongs to A
iii) is independent of $x \in X$

If K is a compact subset of X, then the convergence is uniform over $x \in K$. The function $\Phi: \Sigma \to A$ thus provided is continuous and onto.

Proof:

Let $x \in X$ and $K \in H(X)$, such that $x \in K$.
We construct $\widetilde{K}$ as in Lemma 4. We also define $W: X \to X$ contraction mapping on metric space $(H(X), h(d))$ and we have $A = \lim_{n\to\infty}\{W^{\circ n}(K)\}$, where $W^{\circ n}(K)$ is a Cauchy sequence in $(H(X), h(d))$.
As well as, $\Phi(\sigma,n,x) \in W^{\circ n}(K)$ and it is easy to prove, that if $\lim_{n\to\infty} \Phi(\sigma,n,x)$ exists, then it belongs to A (since for fixed $\sigma \in \Sigma$ the sequence $\{\Phi(\sigma,n,x)\}_{n=1}^{\infty}$ is a Cauchy one) due to Lemma 5, it is conducted $d\big(\Phi(\sigma,m,x),\Phi(\sigma,n,x)\big) \le Ds^{m\wedge n}$, for all $x \in K$, and the right-hand tends to zero as $m,n \to \infty$.
The uniformity of the convergence follows from the fact that the constant D is not dependent of $\in K$.
$\Phi: \Sigma \to A$ is continuous. Indeed, let $\varepsilon > 0$ is given, and we choose n so that $s^n D < \varepsilon$, $\sigma, w \in \Sigma$, such that $d_c(\sigma,w) < \sum_{m=n+2}^{\infty} \frac{N}{(N+1)^m} = \frac{1}{(N+1)^{n+1}}$.
Then we can verify that σ must agree with w through n terms, specifically, that is $\sigma_1, = w_1, \sigma_2, = w_2, \dots, \sigma_n, = w_n$.
So, for each $m \ge n$ we can write

$$d\big(\Phi(\sigma,m,x),\Phi(\sigma,m,x)\big) = d\big(\Phi(\sigma,n,x_1),\Phi(\sigma,n,x_2)\big) \text{, for some pair } x_1,x_2 \in \widetilde{K}.$$

By Lemma 5 the right-hand is smaller than $s^n D$ which is smaller than ε. Taking the limit as $m \to \infty$, we find $d\big(\Phi(\sigma),\Phi(w)\big) < \varepsilon$.
To prove, that Φ is onto, we take $a \in A$, then there is a sequence

$$\{w^{(n)} \in \Sigma: n = 1,2,\dots\}$$

such that $\lim_{n\to\infty} \Phi\big(w^{(n)},n,x\big) = a$. Since (Σ, d_c) is compact it follows that the sequence $\{w^{(n)} \in \Sigma: n = 1,2,\dots\}$ has a convergence subsequence with limit $w \in \Sigma$.
Without loss of generality assume that $\lim_{n\to\infty} w^{(n)} = w$.
If $a(n) =$ the amount of elements of $\{j \in N: w^{(n)} = w_k, for\ 1 \le k \le j\}$, where $N = \{1,2,\dots\}$, then $\lim_{n\to\infty} a(n) = \infty$.
Therefore, $d\Big(\Phi(w,n,x),\Phi\big(w^{(n)},n,x\big)\Big) \le Ds^{a(n)}$, and by taking the limit on both sides as $n \to \infty$, we find $d(\Phi(w),\ \alpha) = 0$, which implies $\Phi(w) = a$, and for that reason $\Phi: \Sigma \to A$ is onto.

Theorem 5.

Let $\{X; w_n: n = 1,2,\dots,N\}$ be a totally disconnected IFS and let $\{A,S\}$ be the associated shift dynamical system.
Let Σ be the associated code space of N symbols and let $T: \Sigma \to \Sigma$ be defined by $T(\sigma) = T(\sigma_1\sigma_2\sigma_3\sigma_4 \dots) = \sigma_2\sigma_3\sigma_4\sigma_5 \dots$ for all $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4 \dots \in \Sigma$.
Then the two dynamical systems $\{A; S\}$ and $\{\Sigma; T\}$ are equivalent. Therefore the Julia set considered as a dynamical system with the Newton- Raphson transformation is chaotic.

We remind the reader that two dynamical systems $\{X_1; f_1\}$ and $\{X_2; f_2\}$ are called equivalent or topologically conjugate, if there is a homomorphism $\theta: X_1 \to X_2$ such that:
$f_1(x_1) = \theta^{-1} \circ f_2 \circ \theta(x_1), for\ all\ x_1 \in X_1$, and
$f_2(x_2) = \theta \circ f_1 \circ \theta^{-1}(x_2), for\ all\ x_2 \in X_2$

All the above analysis leads to Theorem 6.

Theorem 6.

The dynamical system $\{J; f\}$, where J is the Julia set and f is the complex Newton-Raphson transformation, is a chaotic dynamical system.

V. Algorithm Formulation

We now proceed to examine the N-R algorithm behavior at convergence/divergence regions as well as qualitative and quantitive behavior near the Julia Set. In addition, we propose an improved algorithm for constructing equipotential (or contour) lines.

Steps of the iterative algorithm:

- Take three points $z_0^{(n-3)}$, $z_0^{(n-2)}$, $z_0^{(n-1)}$ for the initialization. Each one considered as an initial point of the Newton-Raphson algorithm.
- Assume, inductively that each of three $z_0^{(i)}$ converges after k iterations to the same root p (i.e. to one of the three roots of our example).
- Construct a parabola passing through these 3 points.
- Define a $(n30°)$ the point where the parabola intersects the line passing from p and forms an angle of, say, $30°$, clockwise with the straight line $[p, z_0^{(n-1)}]$.
- Iteratively, the point a $(n30°)$ is an initial point for finding the next initial point of N-R $z_0^{(n)}$ converging after k iterations. This can be found simply by a linear estimation on the line $[p, a(n30°)]$.
- Continuing iteratively, we get $z_0^{(n+1)}$, $z_0^{(n+2)}$ etc, this way obtaining the equipotential line corresponding to k steps of convergence.

It is standard to use different colors corresponding to different roots. Also, to use progressively heavier colors as you get closer to the root (which entails, of course, an increasing smaller number of steps for convergence). Monotonicity here is obvious. It is worth noticing that due to quadratic convergence of Newton-Raphson (when roots are simple), the intensity of each color of the equipotential line "accelerates" as it approaches the root p.

If $a(n30°)$, as initialization, is not "close enough" to $z_0^{(n)}$, we use a smaller angle than 30°, and/or we interpolate the points $z_0^{(i)}$ using cubic splines instead of parabolas. It is helpful to choose angles e.g. 30° which divide 360°.

In more general situations, the complex plane is partitioned into $S \cup A \cup J$, where S is the set of the initial points that converge to some root of the Newton-Raphson algorithm, A is the set of initial points that diverge (to infinity), and J the Julia set (chaotic). In the above example, A is empty. However, the same algorithm can be easily modified to include the general case, $A \neq \emptyset$, simply by counting at each equipotential the number of iterations that exceed a given sphere of radius R.

VI. Conclusions

In this paper, it was proven that the Julia set which derives from the complex Newton-Raphson method implementation is a chaotic dynamical system embedded with the N-R transformation. This is because its topological equivalence with the chaotic shift dynamical system of the Code Space is associated with a shift transformation. The article then presents an iterative algorithmic procedure which defines the Julia set and its graphical representation.

References

[1] Peitgen H.-O., Jurgens H., Saupe D., (2004), "Chaos and Fractals, New Frontiers of Science", 2nd Edition, Springer-Verlag, Germany.
[2] Barnsley M., (1993), "Fractals everywhere", 2nd Edition, Academic Press, UK.
[3] Acheson D., (1997), "From Calculus to Chaos: An Introduction to Dynamics", Oxford University Press, USA.
[4] Falkoner K., (1990), "Fractal Geometry: Mathematical Foundations and Applications", John Wiley &Sons, U.K.
[5] Parker T. S., Chua L., (2011), "Practical Numerical Algorithms for Chaotic Systems", Springer-Verlag, USA.
[6] Pickover C.A., (1998), "Chaos and Fractals A Computer Graphical Journey", Elsevier Science, USA.
[7] Crilly A. J., Eamshaw R. A., Jones H., (1991), "Fractals and Chaos", Springer-Verlag, USA.
[8] Ushiki, S., (1986), "Chaotic Phenomena and Fractal Objects in Numerical Analysis", Patterns and Waves - Qualitative Analysis of Nonlinear Differential Equations, pp. 221–258, doi:10.1016/s0168-2024(08)70132-2.
[9] Lorenz, H.-W., & Nusse, H. E., (2002), "Chaotic attractors, chaotic saddles, and fractal basin boundaries: Goodwin's nonlinear accelerator model reconsidered", Chaos, Solitons & Fractals, 13(5),pp. 957–965, doi:10.1016/s0960-0779(01)00121-7.
[10] Branner B. and Hubbard J., "The iteration of cubic polynomials, part 1: the global topology of parameter space", Acta Math.
[11] Devaney R. L., Keen L., (1989), "Chaos and Fractals: The Mathematics behind the computer graphics", Proceedings of Symposia in Applied Mathematics - American Mathematical Society, Vol. 39.
[12] Smale S., (1987), "On the topology of algorithms, I", 3. of Complexity, vol. 3, pp 81-89.
[13] Smale S., (1988) "The Newtonian contribution to our understanding of the computer", Queen's Quarterly, vol. 95, pp 90-95.
[14] Smale S., (1987), "Algorithms for solving equations", Proceedings of the International Congress of Mathematicians (Amer. Math. Providence), pp 172-195.
[15] Julia G., (1918), "Memoire sur Piteration des functions ratioimelles", J. Math., vol. 8, pp. 47-245.
[16] Peitgen H.-O., Richter P.H., (1986), The Beauty of Fractals, Springer-Verlag, Germany.
[17] Blanchard P., (1984), "Complex analytic dynamics on the Riemann sphere", Bull. Amer. Math. Soc. (N.S.), vol. 11, pp 85-141.
[18] Blanchard P., (1984), "Complex analytic dynamics on the Riemann sphere", Bull. Amer. Math. Soc. (N.S.), vol. 11, pp 85-141.
[19] Douady A. and Hubbard J., (1985), "On the dynamics of polynomial-like mappings", Ann. Sci. Ecole norm. Sup. (4), vol. 18, pp. 287-343.
[20] Fatou P., (1920), "Sur l'iteration des functions transcendentes entieres", Acta Math., vol. 47, pp. 337-370.