GENERALIZED G-SADDLE POINT AND GENERALIZE G-WOLFE DUALITY WITH UNIVEXITY

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Abstract

In this paper, we consider generalized G-Saddle point. Different generalizations of G-Wolfe duality with respective generalized univexity will be derived in the context of Antczak. Strong and weak duality theorems will be derived in this context.

Key words: G-Saddle Point, Duality, Univexity.

1. Introduction

Early in the development of game theory, it was observed that matrix games were equivalent to a dual pain of linear programs (Karlin [17], Charnes [11] and Cottle [12]) Recently, Kawaguchi and Mayurama [18] were formulated dual linear program.

Recently, Corley considered a two-person bi-matrix vector-valued game in which strategy spaces are mixed and introduced the concept of solution of this game. He has also established the necessary and sufficient optimality conditions for the solution of such a game. Chandra and Durga Prasad [10] considered a constrained two-person zero-sum game with vector pay-off and discussed its relation with a pair of multiobjective programming problems. More recently, Singh and Rueda [26] generalized the work of Chandra and Durga Prasad [10] replacing differentiability by sub-differentiability. Also, they were investigated the connection between an equilibrium point of a vector-valued constrained game and a generalized saddle point.

They had shown the relationship between certain convex-concave vector valued games and a pair of nonlinear multiobjective programming problems. Recently, Khan and Hanson [19], has given a new treatment on ratio invexity for a mathematical programming problem. They established sufficient optimality conditions and duality results for an invex fractional programming problem.

2. Notations and Definitions:

In section 2, will discuss basic definitions and notarises, which will needed in the sequel.

Any function $f:R \to R$ is known as strictly increasing if, for all x, y, ε, R, x < y, implies $f(x_1)<(f,y)$

Definitions 2.2 [1] Let S be a non-empty, open subset of R^n and the function $f:X \to R$ be a differentiable function defined on X. If ∃ a differentiable real-valued strictly increasing function on G: If $f:X \to R$ and a vector-valued function $\eta : X \times X \to R^n \ni$ for $X \in X(x \neq n)$, we have $G(f(x))-G(f(u)) \geq G'(f(u))\nabla f(x)\eta^T(x,u)$ then the function f is called G-invex at $u \in X$ on X w.r.t. η.

We will define $d_1-v-\rho-$univexity as follows.

Definition 2.3 [1] Let S be a non-empty, open subset of R^n and the function $f:X \to R$ be a differentiable function defined on X. If ∃ a differentiable real-valued strictly increasing function on G: If $f:X \to R$ and a vector-valued function $\eta : X \times X \to R^n \ni$ for $X \in X(x \neq n)$. Then the function is said to be $d_1-v-\rho-$univexity if.

$$b(x,u)G(f(x))-G(f(u)) \geq G^T(f(x)\nabla f(x)\eta^T(x,u))(+\rho \square \theta\ x,y)\square^2$$

3. Problem formulation

In this section, we consider the following nonlinear mathematical programming problem.

(NP) min $f(x)=(f_1(x),............f_n(x))$

subject to $g_j(x) \leq 0,\ j \in J=\{1,2,.......,m\}$

where S is a non-empty open subject of R^n and $f_i : S \to R$, $j \in J, i \in I$ are in general continuously differentiable function.

In general, set of all feasible solutions is denoted by

$$F_1 = \{x \in S : g_J(x) \le 0, j \in J\}$$

In the sequel, the set of indices of the set of active constraints is denoted by $J_1(u) = \{j \in T : g_j(u) = 0\}$.

Let us consider a suitable definition of the so-called G-Lagrangian function for the chosen nonlinear multi objective mathematical programming problem as:

$$L_G(x; \lambda_1, \mu_1) = \lambda_1 G_{fi}(f_i(x)) + \sum_{J=1}^{m} \mu_j G_{g_j}(g_j(x))$$

where $G_{f_i} : I_f(S) \to R$ and $G_{f_i} : I_{g_i}(S) \to R$ are generally differentiable real valued strictly increasing functions.

Let us introduce a generalized definition of a G-Saddle point for the genralized G-Lagrangian function for the nonlinear multi objective mathematical programming problem.

Definition 3.1: Suppose $(\tilde{x}, \tilde{\lambda}, \tilde{\mu}) \in D \times R + R_+^m$ is a point is called a Generalized G-Saddle point for the nonlinear multi objective mathematical programming problem if the following hold:

i) $L_G(\tilde{x}, \tilde{\lambda}, \mu) \le L_G(\tilde{x}, \tilde{\lambda}, \tilde{\mu}_1)$,

and (ii) $L_G(\tilde{x}, \tilde{\lambda}_1, \tilde{\mu}_1) \le L_G(\tilde{x}, \tilde{\lambda}_1, \tilde{\mu}_1)$,

Let us recall the following Fritz-John necessary optimality conditions for a given mathematical programming problem.

Theorem 3.1 (Necessary optimality conditions According to Antczak[1]

Let $\tilde{x} \in S$ be an optimal point in a mathematical programming problem (MP). Then, the following G-Fritz-John necessary optimality condition are satisfied:

$$\tilde{\lambda}_1, G_f^T\left(f\left(\tilde{x}\right)\right)\nabla f\left(\tilde{x}\right)+\sum_{j=1}^{m}\tilde{\mu}, G_{g_j}^T\left(g_j\left(\tilde{x}\right)\right)\nabla_{g_j}\left(\tilde{x}\right)=0 \quad (3.2)$$

$$\tilde{\mu}_1\left[G_{g_j}\left(g_j\left(x\right)\right)-G_{g_j}\left(g_j\left(\tilde{x}\right)\right)\le 0, j\in J, \forall x\in S\right] \quad (3.3)$$

and $\tilde{\lambda}_1 \ge 0, \tilde{\mu}_1 \ge 0, \left(\tilde{\lambda}_1, \tilde{\mu}_1\right)\ne\left(0,0\right)$ (3.4)

Here G_j is a differentiable real-valued strictly increasing function defined on I_f (S), and G_{g_j} $j\in J_1$, is also a differentiable real-valued strictly increasing function defined on $I_{g_j}\left(S\right)$.

Definition 3.2 : Suppose the equilibrium the point $\left(\tilde{x}, \tilde{\lambda}, \tilde{\mu}_1\right)\in S\times R\times R^n$ satisfies the above G-Fritz-John necessary optimality condition (3.2) – (3.4), then the equilibrium point is known as G-Fritz-John equilibrium fort the nonlinear multi objective mathematical programming problem (MP).

Let us consider the Antczak [1] G-type constraint qualification known as generalized G-type constraint qualification.

Definition 3.3:

The nonlinear multi objective mathematical programming (MP) is said to satisfy the G-type constraint qualification at $\tilde{x}\in S$ if $g_j, j\in J\left(\tilde{x}\right)$, are G_{g_j} –univex w.r. to the same η and θ at $\tilde{x}$ and S and more over $\exists\tilde{x}\in S\ni$

$$G_{g_j}\left(g_j\left(x\right)\right)< G_{g_j}\left(g_j\left(\tilde{x}\right)\right),\ J\in J\left(\tilde{x}\right)$$

According to Antczak [1] we use the following the men in the sequel.

Theorem 3.1

Suppose $\tilde{x}\in S$ is an optimal point in a nonlinear multi objective mathematical problem (MP) and the generalized G-type constraint qualification is to be satisfied at $\tilde{x}$. Then, the following generalized G-K-KK-Tucker necessary optimality conditions are fulfilled.

$$G_f^T\left(f_i\left(\tilde{x}\right)\nabla\right)f\left(\tilde{x}\right)\eta^T\left(x,\tilde{x}\right)+\rho\|\theta\left(x,\tilde{x}\right)\|^2+\sum_{j=1}^{m}\tilde{\mu}, G_{g_j}^T\left(g_j\left(\tilde{x}\right)\right)\times\nabla g_j\left(\tilde{x}\right)=0 \quad (4.5)$$

$$b(x,\tilde{x})\tilde{\mu}_1^1\left[G_{g_j}\left(g_j(x)\right)-G_{g_j}\left(g_j(\tilde{x})\right)\right]\le 0\; j\in J_1,\; x\in S \tag{4.6}$$

and $\tilde{\mu}_1 \ge 0$ (4.7)

hence G_f is a differentiable real – valmel strictly increasing function, which is defined an $I_f(S)$, $G_{g_j}, j\in J$ defined on $I_{g_j}(S)$ and $\sum_{i\in J(\bar{x})}\left[G_{g_j}^T(g_j\tilde{x})\right]^2 \ne 0$.

We will establish the equivalence between the optimal point $\tilde{x}$ and the generalized G-equivalence point. $(\tilde{x},\tilde{\lambda},\tilde{\mu}) \in S\times R\times R^m$ in a nonlinear multi objective programming problem (MP).

Theorem 3.2:

Suppose $(\tilde{x},\tilde{\lambda},\tilde{\mu})\in S\times R_+\times R_+^m$ is a generalized saddle point of the generalized Lagrangian function for the multi objective mathematical programming problem (MP) and the corresponding generalized G-type constraint qualification satisfied at the point $\tilde{x}$.

Further, assume that G_f and G_{g_j} are real-valued strictly increasing function an $I_f(S)$ and $I_{g_j}(S), G_{g_j}(0)=0$ for some $j\in J_1$, and $\sum_{j\in J(g)}\left[G_{g_j}^T\left(g_j(\tilde{x})\right)\right]^2 \ne 0$. Then $\tilde{x}$ is optimal solution in problem (MP).

Proof: This result can be proved by contradiction.

Let us assume that $(\tilde{x},\tilde{\lambda},\tilde{\mu}) \in S\times R_+\times R_+^m$ be a saddle point for the chosen problem (MP).

By definition, we have, for any $\tilde{\mu}_1 \in R_+^m$ $L_G(\tilde{x},\tilde{\lambda},\tilde{\mu}_1)\le L_G(\tilde{x},\tilde{\lambda},\tilde{\mu}_1)$

But, by the definition of the generalized G – Lagrangian function, we set

$$\lambda_1, G_f\left(f_i(x)\right)+\sum_{j=1}^{m}\mu_1 G_{g_j}\left(g_j(x)\right)$$

$$\leq_1 \tilde{\lambda} G_f\left(f_i(\tilde{x})\right)+\sum_{j=1}^{m}\mu_1 G_{g_j}\left(g_j(\tilde{x})\right)$$

which implies that

$$\sum_{j=1}^{m}\mu_1^1\, G_j\left(g_j(x)\right)\leq\sum_{j=1}^{m}\tilde{\mu}_1\, G_{g_j}\left(g_j(\tilde{x})\right)$$

Let us assume that $\mu_1 = 0$ in the above,

we set

$$\sum_{j=1}^{m}\tilde{\mu}_1\, G_{g_j}\left(g_j(\tilde{x})\right)\geq 0 \qquad (3.8)$$

By the feasibility of $\tilde{x}$ in the chosen problem (mp), that follows $g_j(\tilde{x})\leq 0, j\in J_1$ But, by assumption, G_{g_j}, $j\in J_1$ is a differentiable real-valued strictly (monotonic) functions defined the set $I_{g_j}(S)$, and $G_{g_j}(0)=0$ for some $j\in J_1$.

For any $j\in J$ we have

$$G_{g_j}\left(g_j(\tilde{x})\right)\leq G_{g_j}(0)=0$$

But, by $\tilde{\mu}_1, \in R_+^m$, we set

$$\sum_{i=1}^{m}\tilde{\mu}_1 G_{g_j}\left(g_j(\tilde{x})\right)\leq 0 \qquad (3.9)$$

Let us compare equations (3.8) and (3.9),

$$\sum_{i=1}^{m}\tilde{\mu}_1 G_{g_j}\left(g_j(\tilde{x})\right)=0 \qquad (3.10)$$

On the contrary, let $\tilde{x}$ is not optimal in (mp) i.e., $\tilde{x}\in S \ni$

$$f_i(\tilde{x}) < f_i(\tilde{x})\ i=1,2,.....,n.$$

4. Duality Results

Here, we discuss a generalized type duality for the chosen multi objective nonlinear programming. Denoting it as a generalized form as

$$L_G(u,\tilde{\mu}_1) = L_G(u,\tilde{\lambda},\tilde{\mu})$$

Let us consider the corresponding Wolfe type dual is as

$$\text{(G-MWD) max } L_G(u,\tilde{\mu}_1) = G_{f_i}(f_i/u) + \sum_{j=1}^{m} \tilde{\mu}_i\, G_{g_j}(g_j(u))$$

subject to:

$$G_j^1(f_i(u))\nabla f(u)\eta^T(x,u) + \rho\|\theta(x,u)\|^2$$

$$\sum_{j=1}^{m} \tilde{\mu}_i\, G_{g_j}^1(g_j(u)) \times \nabla g_j(u) = 0$$

and $u \in S, \tilde{\mu} \geq 0$

Here G_{f_i} and G_{g_j}, $j \in J_1$ have their usual meanings.

Let us denote the set of feasible solutions as:

$$W_1 = \left\{(u,\tilde{\mu}_1,) \in S \times R^m : G_{f_i}^1(f_i(u))\nabla f(u)\eta^T(x,u) + \sum_{j=1}^{m} \tilde{\mu}_i G_{g_j}(g_j(u), \nabla g_j(u) = 0, \tilde{\mu}_1)\right\}$$

Consequently $\gamma_1 = \{u \in S : (u,\tilde{\mu}_1) \in W_1\}$..

Theorem 4.1 (G-Wolfe weak duality),

Let x and $(u,\tilde{\mu}_1)$ be any feasible solutions in (MFP) and (G-MWD), respectively, moreover, assume that the and fixed $\mu_1^1 \geq 0$, then generalized G-Lagrangian function is $d_1 - \alpha - v - \rho -$univexity at u on FUY w.r. to η and θ, $G_{g_j}(0) = 0$, for $j \in J_1$. Then, $G_{f_i}(f_i(x)) \geq G_{f_i}(f_i(u)) + \sum_{j=1}^{m} \tilde{\mu}_1\ G_{g_j}(g_j(u))$.

Proof:

Let us assume that x and $(x,\tilde{\mu}_1)$ be any two feasible solutions in problem (MP) and (G-MWD), respectively.

By using the assumption that the Langrangian function and by the definition of is $d_1 - v - p -$ univexity at the point u on SUY with respect to η and θ, and also, by using definition 2.2, we obtain

$$G^T f_i\left(f_i(x)\right) + \sum_{j=1}^{m} \tilde{\mu}_1\, G^T g_j\left(g_j(x)\right) - G_{f_i}\left(f_i(u)\right) - \sum_{i=1}^{m} \tilde{\mu}_1 G\, g_j\left(g_j(n)\right)$$

$$\geq \left[G_{f_i}^T \left(f_i(u) \nabla f(u) + \sum_{j=1}^{m} \tilde{\mu}_1 G_g^T g_j\left(g_j(u)\right) \times \nabla g_j(u) \right) \eta^T(x,u) + \rho \Box \theta(x,u) \Box^2 \right]$$

But from the feasibility of $(u,\tilde{\mu}_1)$ in the above problem, we get

$$G_{f_i}\left(f_i(x)\right) + \sum_{i=1}^{m} \tilde{\mu}_1 G_{gj}\left(G_j(x)\right) - G_{f_i}\left(f_i(u)\right) - \sum_{i=1}^{m} \tilde{\mu}_1 G_{g_j}\left(g_j(u)\right) \eta^T(x,u)$$

$$+\rho \Box \theta(x,u) \Box^2 \qquad (4.17)$$

According to the feasibility of x in the chosen problem (MP), it follows that $g_j(x) \leq 0,\ j \in J_1.$

Since $G_{g_j}, j \in J_1$ is a strictly increasing (monotonic) function, which is defined on $I_{g_j}(S)$ and $G_{g_j}(0) = 0,$ for $j \in J_1,$ then, we have

$$G_{g_j}\left(g_j(x)\right) \leq G_{g_j}(0) = 0$$

$\Rightarrow$ by $\tilde{\mu} \geq 0,\ \ j \in T,$ we have

$$\sum_{j=1}^{m} \tilde{\mu}_j G_{g_j}\left(g_j(x) \eta^T(x,y) + \rho \Box \theta(x,y) \Box^2\right) \leq 0$$

From (4.17) and (4.18) one can obtain

$$b(x,y)\left(G_{f_i}\left(f_i(x)\right) \geq G_{f_i}\left(f_i(n)\right)\right) + \sum_{j=1}^{m} \tilde{\mu}_j G_{g_j}\left(g_j(u)\right) \eta^T(x,y) + \rho \Box \theta(x,y) \Box^2$$

Hence the result is proved.

Theorem 4.2 (G-Wolfe Strong Dual)

Suppose $\tilde{\mu}$ is an optimal solution problem (MP) and the generalized G-type constraint qualification is satisfied at $\tilde{\mu}_1$. Now, $\exists \tilde{\mu}_j \in R_+^m \in (\tilde{x}, \tilde{\mu}_j)$ is feasible for (G-MWD) and the corresponding objective functions of the two problem (MP) and (G-MWD) are equal at these points, also, by hypothesis, $(\tilde{x}, \tilde{\mu}_j)$ is optimal in (G-MWD).

Proof: Since $\tilde{x}$ is an optimal solution in (MP) and the generalized G-type constraint qualifications is also satisfied at $\tilde{x}$ then the corresponding (above stated) generalized G-KT necessary optimality conditions are satisfied at $\tilde{x}$.

Then, the corresponding feasibility $(\tilde{x}, \tilde{\mu}_j)$ in the above generalized (G-MWD) follows on from the generalized G-KKT optimality (necessary) condition (3.5) – (3.7)

By using the previous weak duality theorem (4.1), we obtain

$$b(\tilde{x},u)\left(G(f_i\tilde{x}) - G_{fi}(f_i(u)) \geq \sum_{j=1}^{m} \tilde{\mu}_j G_g(g_j(u))\right)\eta^T(\tilde{x},u) + \rho\|\theta(\tilde{x},u)\|^2$$

which holds for all $(u, \tilde{\mu}_j) \in w$

Since $\tilde{x}$ is an optimal solution in the considered problem (MP) and G_{g_j} (0)=0 for some $j \in J_1$, and by applying generalized G-KKT-optimality conditions, one can obtain

$$G_{f_i}(f_i(\tilde{x})) + \sum_{j=1}^{m} \tilde{\mu}_j G_j(g_j(\tilde{x}))\,\eta^T(\tilde{x},u) + \rho\|\theta(\tilde{x},u)\|^2$$

$$\geq G_{f_i}(f_i(u)) + \sum_{j=1}^{m} G_{f_i}(f_i(u)) + \eta^T(\tilde{x},u) + \rho\|\theta(\tilde{x},u)\|^2$$

which holds for all $(u, \tilde{\mu}_j) \in w_1$

Hence, $(\tilde{x}, \tilde{\mu}_j)$ is an optimal in Generalized (G-MWD).

Conclusion

Here we derived different duality theorems for G-Saddle points with respect to generalized d_1-v-ρ-univexity of Antczak type.

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