On a special type of Operator Called (5,2) Jection

Dhananjay Kumar

Research Scholar,
J. P. University, Chapra
Bihar-841301

ABSTRACT

In this article we introduce a new type of operator called (5,2) jection in a linear space. We investigate such operators in C^2 , C being the set of all complex numbers.

Key words : (5,2) jection, projection, trijection, tetrajetion.

1. INTRODUCTION

We are already familiar with the idea of projection. A trijection operator E has been defined by Dr. P. Chandra in his Ph.D. thesis (P.U. 1977) titled "Investigation into the theory of operators and linear spaces" by the relation $E^3= E$ where E is a linear operator on a linear space L. It is a generalisation of projection operator, in the sense that every projection is a trijection but not conversely. Dr. Rajiv Kumar Mishra in his Ph.D. thesis (J.P.U., Chapra 2010) titled 'Study of linear operators and related topic in Functional Analysis" has defined an operator E on a Linear space to be a tetrajection if $E^4 = E$. This also generalises projection operator. These concepts have led me to define E to be a (5,2)-jection if $E^5 = E^2$. Clearly it generalises both idea of projection as well as a tetrajection

Main results

Theorem 1

Let z = (x, y) be an element in C^2 where x, y ∈ C. Let E (z) = (ax + by, cx+dy), a, b, c, d being scalars. We find out conditions for E to be a (5, 2)-jection.

Proof :

By calculation, we find that

$E^2z = E(E(z)) = \{ (a^2 + bc)x + (ab + bd) y, (ac + cd) x + (bc + d^2) y\}$

$= (Ax + By, Cx + Dy)$

where $A = a^2 + bc, B = ab + bd, C = ac + cd, D = bc + d^2$

Replacing E by E^2 , we see that

$E^4z = (A_1x + B_1y, C_1 x + D_1y)$ where

$A_1 = A^2 + BC$, $B_1 = B(A+D)$, $C_1 (A+D)$, $D_1 = BC + D^2$

Hence,

$E^5 z = E(E^4 z) = [a(A_1x + B_1 y) + b(C_1 x + D_1 y), c(A_1 x + B_1 y) + d(C_1 x + D_1 y)]$

$=[(aA_1 + bC_1)x + (aB_1 + bD_1)y, (cA_1 + dC_1)x + (cB_1 + dD_1)y]$

Hence if $E^5 = E^2$, then $E^5 z = E^2 z$ and comparing R.H.S of both,

$aA_1 + bC_1 = A$, $aB_1 + bD_1 = B$, $cA_1 + dC_1 = C$, $cB_1 + dD_1 = D$.

Now

$aA_1 + bC_1 = A \Rightarrow a(A^2 + BC) + bC(A + D) = A$ ....(1)

$aB_1 + bD_1 = B \Rightarrow aB(A+D) + b(BC+D^2) = B$ ....(2)

$cA_1 + dC_1 = C \Rightarrow c(A^2 + BC) + dC(A+D) = C$ ....(3)

$cB_1 + dD_1 = D \Rightarrow cB(A+D) + d(BC + D^2) = D$ ....(4)

Equations (1) to (4) are required conditions.

Theorem 2

Let ad = bc.Show that E is a (5,2)-jection if $A = a^2 + bc$ is either 0 or a or $a\omega$ or $a\omega^2$, ω being a cube root of unity.

Proof:-

Since ad=bc, we have $d = \frac{bc}{a}$ (assume $a \neq 0$)

Then $A = a^2 + bc = a^2 + ad = a(a+d)$. so $a+d = \frac{A}{a}$

$B = b(a+d) = \frac{bA}{a}$, $C = c(a+d) = \frac{cA}{a}$

$D = bc + d^2 = ad + d = d(a+d) = \frac{dA}{a}$

then $A_1 = A^2 + BC = A^2 + \frac{bcA^2}{a^2} = A^2\left(\frac{a^2+bc}{a^2}\right) = \frac{A^3}{a^2}$.

$B_1 = B(A+D) = \frac{bA}{a}\left(A + \frac{dA}{a}\right) = \frac{bA}{a^2}(aA + dA) = \frac{bA^2}{a^2} \bullet \frac{A}{a} = \frac{bA^3}{a^3}$

$C_1 = C(A+D) = \frac{cA}{a}\left(A + \frac{dA}{a}\right) = \frac{ca^2}{a^2} \bullet \frac{A}{a} = \frac{cA^3}{a^3}$

$D_1 = BC+D^2 = bc\frac{A^2}{a^2} + \frac{d^2A^2}{a^2} = \frac{A^2}{a^2}(bc + d^2) = \frac{A^2}{a^2}\bullet\frac{dA}{a} = \frac{dA^3}{a^3}$

so due to (1),

$$\frac{aA^3}{a^2} + bc\frac{A^3}{a^3} = A \Rightarrow \frac{A^3}{a^3}(a^2 + bc) = A$$

$$\Rightarrow \frac{A^3}{a^3}\bullet A = A \quad \Rightarrow A(1- \frac{A^3}{a^3}) = 0$$

$$\Rightarrow A = 0 \text{ or } A^3 = a^3$$

$$\Rightarrow A = 0 \text{ or } a \text{ or } \omega a \text{ or } \omega^2 a$$

due to (2)

$$ab\frac{A^3}{a^3} + bd\frac{A^3}{a^3} = \frac{bA}{a} \Rightarrow ab\, A^3 + bd\, A^3 = ba^2A$$

$$bA(aA^2 + dA^2) = bA^2 \Rightarrow bA[(a+d)A^2 - a^2] = 0$$

$$\Rightarrow b=0 \text{ or } A=0 \text{ or } (a + d)A^2 = a^2 \text{ i.e } \frac{A^3}{a^3} = 1$$

$$\Rightarrow b=0 \text{ or } A = 0, a, \omega a, \omega^2 a.$$

due to (3)

$$c\frac{A^3}{a^2} + dc\frac{A^3}{a^3} = \frac{cA}{a} \Rightarrow acA^3 + cdA^3 = cAa^2$$

Hence as in (2), $c=0$ or $A=0, a, \omega a, \omega^2 a$.

due to (4)

$$cb\frac{A^3}{a^3} + d^2\frac{A^3}{a^3} = \frac{dA}{a} \Rightarrow (bc + d^2)A^3 = da^2A$$

$$\Rightarrow \frac{dA}{a}\bullet A^3 = da^2A \Rightarrow dA(A^3 - a^3) = 0$$

$$\Rightarrow d=0 \text{ or } A = 0, a, \omega a, \omega^2 a.$$

Hence considering all 4 equations, common solution is

$A = 0, a, \omega a, \omega^2 a$.

Theorem 3

Let ad = bc and A=0.we show that $E^2 = 0$ and we give a few examples of E in this case.

Proof :-

we are given that A= 0

so $a^2+bc = 0 \Rightarrow d = \frac{bc}{a} = -a \Rightarrow a+d = 0$.

Also $c = -\frac{-a^2}{b}$ (b≠0) and $b = -\frac{-a^2}{c}$ (c≠0)

Then B = b(a+d) =0,C = c(a+d)=0

$D = bc + d^2 = ad + d^2 = d(a+d)=0$.

so in this case,

$E(x,y)= (ax+by,-\frac{-a^2x}{b} - ay)$ if b ≠0

and $E(x,y)= (ax- \frac{a^2y}{c}, cx-ay)$ (c≠ 0)

Also $E^2(x,y)=(Ax+By,Cx+Dy)=(0,0)$.

Thus $E^2 = 0$. clearly $E^5 = 0=E^2$

We consider some examples in this case

Let a=0,then E(x,y)=(by,0).

Let a=1,then $E(x,y)=(x+by, \frac{-x}{b} - y)$ (b ≠ 0)

$=(x - \frac{y}{c}, cx - y)$ (c ≠0)

Further if c =1 then E(x,y)=(x-y,x-y).

Let a =1,b=1 then E(x,y) =(x+y,-x-y).

when b=1,then $E(x,y) = (ax+y,-a^2x-ay)$

when b=ω, $E(x,y) =(ax+\omega y,-a^2\omega^2x - ay)$

if also a=1, $E(x,y)=(x+\omega y,-\omega^2x-y)$

when $b = \omega^2$, $E(x,y)=(ax+\omega^2y,-a^2\omega x-ay)$

similarly we may get some further examples.

Theorem 4

Let ad= bc and A=a($\neq$0). We show that E is a projection and discuss a few examples in this case.

Proof:-

In this case $a^2 + bc = a \Rightarrow bc = a - a^2$

$d = \frac{bc}{a} = \frac{a-a^2}{a} = 1-a$.Hence a+d = 1.

so $E(x,y)=(ax+by, \frac{a-a^2}{b}x + (1-a)y)$ (b$\neq$0)

$(ax+ \frac{a-a^2}{c}y, cx+(1-a)y)$ (c$\neq$ 0)

Also A=a, B=b(a+d) = b, C = c(a+d)=c

$D=bc+d^2 = ad + d^2 = d(a+d)=d$.

Hence $E^2(x,y)=(Ax+By,Cx+Dy)=(ax+by,cx+dy)=E(x,y)$

i.e. $E^2 = E$ or E is a projection.

we consider a few examples in this case.

Let a=1, then E(x, y)= (x+by,0)

if b=0 then E(x,y)=(x,0)

if a+b =1 then b= 1-a=d

since $bc = a-a^2 = a(1-a) = ab$, we have c=a if b$\neq$0.

Hence E(x,y)=(ax+by, ax+by), (b$\neq$0).

in particular if $a=b=\frac{1}{2}$,then $E(x,y)=(\frac{x+y}{2},\frac{x+y}{2})$

Theorem 5

let ad=bc and A=ωa. We show that $E^2=\omega E$ and consider a few examples.

proof:-

Here $a^2+ bc = \omega a \Rightarrow \omega a - a^2$

$d=\frac{bc}{a}=\frac{\omega a-a^2}{a}=\omega\text{-}a \Rightarrow a+d=\omega \quad (a\neq 0)$

so, $E(x,y)=(ax+by, \frac{\omega a-a^2}{b}x+(\omega\text{-}a)y) \quad (b\neq 0)$

$(ax+\frac{\omega a-a^2}{c}y, cx+(\omega\text{-}a)y) \qquad (c\neq 0)$

Also $B=b(a+d)=b\omega$, $C=c(a+d)=c\omega$

$D=d(a+d)=d\omega$.

Hence $E^2(x, y) = (\omega ax + \omega by, \omega cx + \omega dy) = \omega(ax + by, cx+ dy) = \omega E(x, y)$

So $E^2=\omega E$ and E is not a projection.

Hence $E^3=\omega E^2=\omega^2 E$

$E^4= \omega.\omega^2 E=E \Rightarrow E^5 = E^2$

so E is a tetrajection also.Now we discuss some examples

Let a=0 then $E(x,y)=(by,\omega y)$

Let $a=\omega$ then $E(x,y)=(\omega x+by,0)$

Let $a= \omega^2$ then $E(x,y)=(\omega^2 x +by,\frac{1-\omega}{b}x+(\omega\text{-} \omega^2)y)$

Let $a= \omega=b$ then $E(x,y)=(\omega x+ \omega y,0)$

Let $a= \omega, b= \omega^2$ then $E(x,y)=(\omega x+ \omega^2 y,0)$

Let $a= \omega, b=0$ then $E(x,y)=(\omega x,cx)$

Theorem 6

Let ad = bc and $A= \omega^2 a$.We show that $E^2 = \omega^2 E$ and consider a few examples.

Proof:-

In this case $a^2 +bc = \omega^2 a \Rightarrow bc = \omega^2 a - a^2 \Rightarrow c = \frac{\omega^2 a-a^2}{b} \quad (b\neq 0)$

Also $b = \frac{\omega^2 a-a^2}{c} \quad (c \neq 0)$.

Also $d = \frac{bc}{a} =\frac{\omega^2 a-a^2}{a} = \omega^2 -a \Rightarrow a+d =\omega^2$.

Hence $E(x,y) = (ax+by, \frac{\omega^2 a-a^2}{b}x + (\omega^2 - a)y) \quad (b\neq 0)$

$(ax + (\frac{\omega^2 a - a^2}{c}, cx + (\omega^2 - a)y)$ $(c \neq 0)$

In this case $B = b(a+d) = b\omega^2$, $c = c\,\omega^2$,

$D = bc + d^2 = \omega^2 a - a^2 + (\omega^2 - a)^2 = \omega^2 a - a^2 + \omega - 2\omega^2 a + a^2$

$= \omega - \omega^2 a = \omega^2(\omega^2 - a) = \omega^2 d$

Hence $E^2(x,y) = (\omega^2 ax + \omega^2 by, \omega^2 cx + \omega^2 dy) = \omega^2 E(x, y)$

$\Rightarrow E^2 = \omega^2 E$.

clearly E is not a projection

Also $E^3 = \omega^2 E^2 = \omega E$, $E^4 = \omega E^2 = E$, $E^5 = E^2$

Thus E is a tetrajection,as well as a(5,2)-jection.

Let us consider a few examples

let $a = \omega^2$ then $E(x, y) = (\omega^2 x + by, 0)$ $(b \neq 0)$

$= (\omega^2 x, cx)$ $(c \neq 0)$

let $a = \omega$ then $E(x,y) = (\omega x + by, \frac{1-\omega^2}{b} x + (\omega^2 - \omega)y)$ $(b \neq 0)$

If further $b = \omega$ then $E(x,y) = (\omega x + \omega y, (\omega^2 - \omega)x + (\omega^2 - \omega)y)$

$= (\omega(x+y), (\omega^2 - \omega)(x+y)), = (x+y)(\omega, \omega^2 - \omega)$.

Theorem 7

Let $bc = ad$, then

we come to the case when b=0 or c =0 . we also consider same examples

Proof:-

Since $bc = ad$ we have ad=0.so at least one of a,d is 0or both are 0.Take the case of b=0.

So we consider two cases (i) b=0, a=0 or (ii) b=0,d=0

Let $a = 0$, $b = 0$ then

Let $A=0$, $B=0$, $C=cd$, $D=d^2$.

in theorem (1),(1) and (2) are obvious.

Due to (3), $cd^4 - cd = 0 \Rightarrow cd(d^3 - 1) = 0$

$\Rightarrow c=0,\ d=0,\ 1,\ \omega,\ \omega^2$

If we take a=b=c=d=0,we get E=0,zero operator.

Taking a=b=c=0,d=1 we get E(x,y) =(0,y), a projection.

If a=b=c=0, d=ω we get E(x,y)= (0,ωy),a tetrajection.

If a=b=c=0, $d=\omega^2$ we get E(x,y)= $(0,\omega^2 y)$,a tetrajection.

If a=b=0, c≠0, d=1,we get E(x,y)= (0,cx),for which $E^2=0$.

If a=b=0, c≠0, d=o, we get E(x,y) = (0,cx+y), a projection.

If a=b=0, c≠0,d=ω,we get E(x,y) = (0,cx+ωy), a tetrajection.

If a=b=0, c≠0,d = ω^2,then E(x,y) = $(0,cx+\omega^2 y)$

Now come to the case when b=d =0.

Hence A = a^2,B =0,C =ac,D=0.

Due to (1), $a^5 = a^2 \Rightarrow a=0,1,\omega,\omega^2$

(2) gives 0 =0 .(3) gives $ca^4 = ac \Rightarrow ca(a^3 - 1)=0$

$\Rightarrow\ c=0,a=0,\ 1,\omega,\omega^2.$

Due to (4),0 =0.

If b=c =d =0,then E(x,y)= (ax,0) where $a=0,1,\omega,\omega^2$

Thus E(x,y)= (0,0) or (x,0), a projection.

or E(x,y)= (ωx,0),a tetrajection.

or E(x,y) = $(\omega^2 x,0)$, a tetrajection.

Let b=d=0 but c≠0 then a =0, $1,\omega,\omega^2$

So we have E(x,y) = (o,cx).Then $E^2 = 0$.

We also have E(x,y)= (x,cx), a projection.

also E(x,y) = (ωx,cx)

and E(x,y) = $(\omega^2 x,cx)$, a tetrajection.

Case with c=0 can be similarly dealt with.

References

(1) Chandra, P. Investigation into the theory of operators and linear spaces ph. D Thesis,Patna University,1977

(2) Chandra, P. and Tripathy, B.C. "On generalised Kothe-Toeplitz duals of some sequence spaces '' , Indian J. of pure and applied Math. 33(8) (2002),1301-1306.

(3) Mishra, R. K.'' study of linear operators and related topics in Functional Analysis'' Ph.D Thesis, J. P . University, chapra, 2010.

(4) Singh, L. B.'' Some Advancements in the theory of Linear Spaces and Linear Operators'' , Ph.D Thesis, Patna University, 1989

(5) Rath, D. and Tripathy, B.C.,'' Characterisation of certain Matrix Operators '' , J. orissa Math soc.8(1989),121-134

(6) Chandra, P.:''On dual spaces of some sequence spaces'', International Journal of Mathematical Sciences, BHU, Varanasi. Vol .IV ,No. 1,117-121(June 2005).