New techniques for solving 2nd order ordinary differential equations

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Abstract: Differential Equation $f(D)y = R(x)$ where $f(D) = (D+m)^2$ $m \in R$ and we assume that $R(x)$ is cos nx OR sin nx where n∈R. The general solution of above D.E. like $(D+m)^2 = \cos nx$ and $(D+m)^2 = \sin nx$ is of the form $G.S. = C.F. + P.I.$ Where C.F. is obtain from usual way and

(i) $\mathbf{P.I.} = \left[\frac{2mn\sin nx - [n^2 - m^2]\cos nx}{(2mn)^2 + [n^2 - m^2]}\right]$ **for cosine function and** **(ii)** $\mathbf{P.I.} = -\left[\frac{2mn\cos nx + [n^2 - m^2]\sin nx}{(2mn)^2 + [n^2 - m^2]}\right]$ **for sine function.**

Index Terms - Differential Equation, Order & Degree of differential equation, General Solution, Particular Solution, Complementary Function, Particular Integral**.**

I. Introduction

Differential Equation (D.E.):
An equation involving differential equation and independent and dependent variable is called a differential equation. If a differential involves ordinary coefficients only, it is called ordinary differential equation and if it involves partial derivatives it is called partial **Differential Equation**.

Order & Degree of D.E.:
The order of differential equation is the order of highest derivative in given differential equation.
Degree of differential equation is the degree of highest derivative after removing fractions & radicals.

General Solution (G.S.):
"If y_1 & y_2 are two solution of the differential equation $\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_n y = 0$ .... (1)
Then $c_1y_1 + c_2y_2$ is also its solution where c_1 and c_2 are arbitrary constant." In general, the general solution differential equation of order n will have n arbitrary constants, therefore it follow that if $y_1, y_2, \ldots, y_n$ are n independent solution of the eq^n(1) then $u = c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is the complete **General Solution** of eq^n(1).

Particular Solution:
The solution obtained from the General Solution by giving particular values to the arbitrary constant is called **Particular Solution**.

Complementary Function (C.F.):
The differential equation under consideration is $\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_n y = 0$ Which, in symbolic operator form, is $\boldsymbol{f(D)y = 0}$.Sinice $f(D)$ is a polynomial in D of degree n and D behaves the same as an algebraic quantity we can, in general, factorize $f(D)$ into n linear factors and $\boldsymbol{f(D)y = 0}$ can be written as $(D - m_1)(D - m_2)\ldots(D - m_n)y = 0$ Where $m_1, m_2, \ldots, m_n$ are the root of the **Auxiliary Equation(A.E.)** $f(m) = 0$.

Particular Integral (P.I.):
Before discussing ways to find Particular Integral (P.I.) we have to introduce inverse operator $\frac{1}{f(D)}$.
$\frac{1}{f(D)}X(x)$ is that function of x which when operated upon by $f(D)$ gives $X(x)$ only. Therefore $\frac{1}{f(D)}X$ satisfies the equation $f(D)y = X$ and hence it is the **Particular Integral (P.I.)** of equation $f(D)y = X$

and is symbolically written as $Yp = \frac{1}{f(D)} X$. For the Above definition General Solution (G.S.), Complementary Function (C.F.), Particular Solution (P.I.) is relation of $\boldsymbol{G.S. = C.F. + P.I.}$

2. Known Results:-

(A) $(D+3)^2 = \cos 3x$

- **By General Method:-**

$$\text{P.I.} = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{(D+3)^2} \cos(3x)$$

$$= \frac{1}{(D+3)(D+3)} \cos(3x)$$

$$= \frac{1}{(D)+3)} \left[\frac{1}{(D+3)} \cos(3x)\right]$$

$$= \frac{1}{(D+3)} \left[e^{-3x} \int e^{3x} \cos(3x)\, dx\right]$$

$$= \frac{1}{D+3} \left[[e^{-3x} * \frac{e^{3x}}{18} [3\cos(3x) + 3\sin(3x)]\right]$$

$$= \frac{1}{D+3} [\frac{1}{6} [\sin(3x) + \cos(3x)]]$$

$$= \frac{1}{6} [\frac{1}{D+3} \sin(3x) + \frac{1}{D+3} \cos(3x)]$$

$$= \frac{1}{6} \left[e^{-3x} \int e^{3x} \sin(3x)\, dx + e^{-3x} \int e^{3x} \cos(3x)\, dx\right]$$

$$= \frac{1}{6} \left[\frac{1}{18} [3\sin(3x) - 3\cos(3x)] + \frac{1}{18} [3\cos(3x) + 3\sin(3x)]\right]$$

$$= \frac{1}{36} [\sin(3x) - \cos(3x) + \cos(3x) + \sin(3x)]$$

$$= \frac{2\sin(3x)}{36}$$

$$= \frac{\sin(3x)}{18}$$

- **By Shortcut Method:-**

$$\text{P.I} = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{(D+3)^2} \cos(3x)$$

$$= \frac{1}{D^2 + 6D + 9} \cos(3x)$$

$$= \frac{1}{6D} \cos(3x)$$

$$= \frac{1}{6}\int \cos(3x)$$

$$= \frac{1}{6}\frac{\sin(3x)}{3}$$

$$= \frac{\sin(3x)}{18}$$

(B) $(D+3)^2 = \sin(3x)$

- **By General Method:-**

$$\text{P.I.} = \frac{1}{f(D)}R(x)$$

$$= \frac{1}{(D+3)(D+3)}\sin(3x)$$

$$= \frac{1}{D+3}\left[\frac{1}{D+3}\sin(3x)\right]$$

$$= \frac{1}{D+3}\left[e^{-3x}\int e^{3x}\sin(3x)\,dx\right]$$

$$= \frac{1}{D+3}\left[\frac{e^{-3x}e^{3x}}{18}[3\sin(3x) - 3\cos(3x)]\right]$$

$$= \frac{1}{6}\frac{1}{D+3}[\sin(3x) - \cos(3x)]$$

$$= \frac{1}{6}\left[\frac{1}{D+3}\sin(3x) - \frac{1}{D+3}\cos(3x)\right]$$

$$= \frac{1}{6}\left[e^{-3x}\int e^{3x}\sin(3x)\,dx - e^{-3x}\int e^{3x}\cos(3x)\,dx\right]$$

$$= \frac{1}{6}\left[\frac{1}{18}[5\sin(3x)3\cos(3x)] - \frac{1}{18}[3\cos(3x) + 3\sin(3x)]\right]$$

$$= \frac{1}{36}[\sin(3x) - \cos(3x) - \cos(3x) - \sin(3x)]$$

$$= \frac{1}{36}[-2\cos(3x)\,]$$

$$= -\frac{\cos 3x}{18}$$

- **By Shortcut Method:-**

$$\text{P.I} = \frac{1}{f(D)}R(x)$$

$$= \frac{1}{(D+3)^3}\sin(3x)$$

$$= \frac{1}{D^2+6D+9}\sin(3x)$$

$$= \frac{1}{6D}\sin(3x)$$

$$= \frac{1}{6}\int \sin(3x)\,dx$$

$$= -\frac{\cos(3x)}{18}$$

4. Main Result

1.) $(D+m)^2 = \cos nx \quad$ where $m, n \in R$

$$P.I = \frac{2mn\sin nx - (n^2-m^2)\cos nx}{(2mn)^2 + (n^2-m^2)^2}$$

2.) $(D+m)^2 = \sin nx \quad$ where $m, n \in R$

$$P.I = -\left[\frac{2mn\cos nx - (n^2-m^2)\sin nx}{(2mn)^2 + (n^2-m^2)^2}\right]$$

❖ **Examples:-**

1.) $(D+2)^2 = \cos 3x$

➢ **By Shortcut Method:**

$$P.I = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{D^2+4D+4}\cos(3x)$$

$$= \frac{1}{4D-5}\cos(3x)$$

$$= \frac{4D+5}{16D^2-25}\cos(3x)$$

$$= \frac{4D+5}{-169}\cos(3x)$$

$$= \frac{12\sin(3x)}{169} - \frac{5\cos(3x)}{169}$$

By New Derived Results:

$m = 2$ and $n = 3$

$$P.I = \frac{2(2)(3)\sin(3x) - [(3)^2-(2)^2]\cos(3x)}{[2(2)(3)]^2 + [(3)^2-(2)^2]}$$

$$= \frac{12\sin(3x) - 5\cos(3x)}{169}$$

$$= \frac{12\sin(3x)}{169} - \frac{5\cos(3x)}{169}$$

2.) $(D+2)^2 = \cos(-3)x$

➢ **By Shortcut Method:**

$$P.I = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{D^2+4D+4}\cos(-3x)$$

$$= \frac{1}{4D-5}\cos(3x)$$

$$= \frac{4D+5}{16D^2-25}\cos(3x)$$

$$= \frac{4D+5}{-169}\cos(3x)$$

$$= \frac{12\sin(3x)}{169} - \frac{5\cos(3x)}{169}$$

By New Derived Results:

$m = 2$ and $n = -3$

$$P.I = \frac{2(2)(-3)\sin(-3x) - [(-3)^2-(2)^2]\cos(-3x)}{[2(2)(-3)]^2 + [(-3)^2-(2)^2]}$$

$$= \frac{12\sin(3x) - 5\cos(3x)}{169}$$

$$= \frac{12\sin(3x)}{169} - \frac{5\cos(3x)}{169}$$

3.) $(D-2)^2 = \cos 3x$

➢ **By Shortcut Method:**

$$P.I = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{D^2-4D+4}\cos(3x)$$

$$= \frac{1}{-4D-5}\cos(3x)$$

By New Derived Results:

$m = -2$ and $n = 3$

$$P.I = \frac{2(-2)(3)\sin(3x) - [(3)^2-(-2)^2]\cos(3x)}{[2(2)(3)]^2 + [(3)^2-(2)^2]}$$

$$= \frac{-12\sin(3x) - 5\cos(3x)}{169}$$

$$= -\frac{4D-5}{16D^2-25}\cos(3x)$$

$$= -\frac{4D+5}{-169}\cos(3x)$$

$$= -\frac{12\sin(3x)}{169} - \frac{5\cos(3x)}{169}$$

$$= -\frac{12\sin(3x)}{169} - \frac{5\cos(3x)}{169}$$

4.) $(D-2)^2 = \cos(-3)x$

- **By Shortcut Method:**

$$P.I = \frac{1}{f(D)}R(x)$$

$$= \frac{1}{D^2-4D+4}\cos(-3x)$$

$$= \frac{1}{-4D-5}\cos(3x)$$

$$= -\frac{4D-5}{16D^2-25}\cos(3x)$$

$$= -\frac{4D+5}{-169}\cos(3x)$$

$$= -\frac{12\sin(3x)}{169} - \frac{5\cos(3x)}{169}$$

By New Derived Results:

$m = -2$ and $n = -3$

$$P.I = \frac{2(-2)(-3)\sin(-3x) - [(-3)^2-(-2)^2]\cos(-3x)}{[2(2)(3)]^2+[(3)^2-(2)^2]}$$

$$= \frac{12\sin(-3x)-5\cos(-3x)}{169}$$

$$= -\frac{12\sin(3x)}{169} - \frac{5\cos(3x)}{169}$$

5.) $(D+2)^2 = \sin 3x$

- **By Shortcut Method:**

$$P.I = \frac{1}{f(D)}R(x)$$

$$= \frac{1}{D^2+4D+4}\sin(3x)$$

$$= \frac{1}{4D-5}\sin(3x)$$

$$= \frac{4D+5}{16D^2-25}\sin(3x)$$

$$= \frac{4D+5}{-169}\sin(3x)$$

$$= -\left[\frac{12\cos(3x)}{169} + \frac{5\sin(3x)}{169}\right]$$

By New Derived Results:

$m = 2$ and $n = 3$

$$P.I = -\left[\frac{2(2)(3)\cos(3x) + [(3)^2-(2)^2]\sin(3x)}{[2(2)(3)]^2+[(3)^2-(2)^2]}\right]$$

$$= -\left[\frac{12\cos(3x)+5\sin(3x)}{169}\right]$$

$$= -\left[\frac{12\cos(3x)}{169} + \frac{5\sin(3x)}{169}\right]$$

6.) $(D+2)^2 = \sin(-3)x$

- **By Shortcut Method:**

$$P.I = \frac{1}{f(D)}R(x)$$

$$= \frac{1}{D^2+4D+4}\sin(-3x)$$

$$= \frac{1}{4D-5}\sin(-3x)$$

$$= \frac{4D+5}{16D^2-25}\sin(-3x)$$

By New Derived Results:

$m = 2$ and $n = -3$

$$P.I = -\left[\frac{2(2)(-3)\cos(-3x) + [(-3)^2-(2)^2]\sin(-3x)}{[2(2)(-3)]^2+[(-3)^2-(2)^2]}\right]$$

$$= -\left[\frac{-12\cos(-3x)+5\sin(-3x)}{169}\right]$$

$$= \frac{12\cos(3x)}{169} + \frac{5\sin(3x)}{169}$$

$$= \frac{4D+5}{-169}\sin(-3x)$$

$$= \frac{12\cos(3x)}{169} + \frac{5\sin(3x)}{169}$$

7.) $(D-2)^2 = \sin 3x$

- **By Shortcut Method:**

$$P.I = \frac{1}{f(D)}R(x)$$

$$= \frac{1}{D^2-4D+4}\sin(3x)$$

$$= \frac{1}{-4D-5}\sin(3x)$$

$$= \frac{-4D+5}{16D^2-25}\sin(3x)$$

$$= \frac{-4D+5}{-169}\sin(3x)$$

$$= \frac{12\cos(3x)}{169} - \frac{5\sin(3x)}{169}$$

By New Derived Results:

$m = -2$ and $n = 3$

$$P.I = -\left[\frac{2(-2)(3)\cos(3x)+[(3)^2-(-2)^2]\sin(3x)}{[2(-2)(3)]^2+[(3)^2-(-2)^2]}\right]$$

$$= -\left[\frac{-12\cos(3x)+5\sin(3x)}{169}\right]$$

$$= \frac{12\cos(3x)}{169} - \frac{5\sin(3x)}{169}$$

8.) $(D-2)^2 = \sin(-3)x$

- **By Shortcut Method:**

$$P.I = \frac{1}{f(D)}R(x)$$

$$= \frac{1}{D^2-4D+4}\sin(-3x)$$

$$= \frac{1}{-4D-5}\sin(-3x)$$

$$= \frac{-4D+5}{16D^2-25}\sin(-3x)$$

$$= \frac{-4D+5}{-169}\sin(-3x)$$

$$= -\frac{12\cos(3x)}{169} + \frac{5\sin(3x)}{169}$$

By New Derived Results:

$m = -2$ and $n = -3$

$$P.I = -\left[\frac{2(-2)(-3)\cos(-3x)+[(-3)^2-(-2)^2]\sin(-3x)}{[2(-2)(3)]^2+[(3)^2-(-2)^2]}\right]$$

$$= -\left[\frac{12\cos(3x)-5\sin(3x)}{169}\right]$$

$$= \frac{-12\cos(3x)}{169} + \frac{5\sin(3x)}{169}$$

5. Mathematical Derivation

➔For COSINE

$$(\mathbf{D+m})^2 = \mathbf{\cos nx}$$

$$\mathbf{P.I.} = \frac{\mathbf{2mn\sin nx - [n^2-m^2]\cos nx}}{\mathbf{(2mn)^2-(n^2-m^2)^2}}$$

Step 1:- $n = 1 = m$

$$(D+1)^2 = \cos x$$

- **By Shortcut Method:**

$$P.I = \frac{1}{f(D)}R(x)$$

By New Derived Results:

$m = 1$ and $n = 1$

$= \frac{1}{D^2+2D+1}\cos(x)$

$= \frac{1}{2D}\cos(x)$

$= \frac{1}{2}\sin(x)$

$P.I = \frac{2(1)(1)\sin(x)-[(1)^2-(1)^2]\cos(x)}{[2(1)(1)]^2+[(1)^2-(1)^2]}$

$= \frac{2\sin(x)}{4}$

$= \frac{1}{2}\sin(x)$

Step 2:- $m = k = n$

To suppose $(D+k)^2 = \cos kx$

- **By Shortcut Method:**

$P.I = \frac{1}{f(D)}R(x)$

$= \frac{1}{D^2+2kD+k^2}\cos(kx)$

$= \frac{1}{2kD}\cos(kx)$

$= \frac{1}{2k^2}\sin(kx)$

By New Derived Results:

$m = 1$ and $n = 1$

$P.I = \frac{2(k)(k)\sin(kx)-[(k)^2-(k)^2]\cos(kx)}{[2(k)(k)]^2+[(k)^2-(k)^2]}$

$= \frac{2k^2\sin(kx)}{4(k^2)^2}$

$= \frac{1}{2k^2}\sin(kx)$ ------ **(1)**

Step 3:- To Prove $m = k+1 = n$

$(D+k+1)^2 = \cos(k+1)x$

Suppose k+1=t

k=t-1

So, $(D+t)^2 = \cos tx$

$P.I = \frac{2(t)(t)\sin(tx)-[(t)^2-(t)^2]\cos(tx)}{[2(t)(t)]^2+[(t)^2-(t)^2]}$

$= \frac{2(t)^2\sin(tx)}{4(t^2)^2}$

$= \frac{\sin tx}{2t^2}$ $(\because$ **from** $\mathbf{eq^n(1)})$

$= \frac{\sin(k+1)x}{2(k+1)^2}$

➔ **<u>For</u> <u>SINE</u>:-**

$\mathbf{(D+m)^2 = \sin nx}$

$\mathbf{P.I.} = -\left[\frac{\mathbf{2mn\cos nx - [n^2-m^2]\sin nx}}{\mathbf{(2mn)^2-(n^2-m^2)^2}}\right]$

Step 1:- $n = 1 = m$

$(D+1)^2 = \sin x$

- **By Shortcut Method:**

$P.I = \frac{1}{f(D)}R(x)$

$= \frac{1}{D^2+2D+1}\sin(x)$

By New Derived Results:

$m = 1$ and $n = 1$

$P.I = -\left[\frac{2(1)(1)\cos(x)+[(1)^2-(1)^2]\sin(x)}{[2(1)(1)]^2+[(1)^2-(1)^2]}\right]$

$= \frac{1}{2D}\sin(x)$ $\qquad = -\frac{2\cos(x)}{4}$

$= -\frac{1}{2}\cos(x)$ $\qquad = -\frac{1}{2}\cos(x)$

Step 2:- $m = k = n$

To suppose $(D+k)^2 = \sin kx$

- **By Shortcut Method:**

$$P.I = \frac{1}{f(D)}R(x)$$

$$= \frac{1}{D^2+2kD+k^2}\sin(x)$$

$$= \frac{1}{2kD}\sin(x)$$

$$= -\frac{1}{2k}\cos(x)$$

By New Derived Results:

$m = 1$ and $n = 1$

$$P.I = -\left[\frac{2(k)(k)\cos(kx)+[(k)^2-(k)^2]\sin(kx)}{[2(k)(k)]^2+[(k)^2-(k)^2]}\right]$$

$$= -\frac{2k^2\cos(x)}{4(k^2)^2}$$

$$= -\frac{1}{2}\cos(x) \qquad \text{------} \textbf{(2)}$$

Step 3:- To Prove $m = k + 1 = n$

$(D+k+1)^2 = \sin(k+1)x$

Suppose k+1=t

k=t-1

So, $(D+t)^2 = \sin tx$

$$P.I = -\left[\frac{2(t)(t)\cos(tx)+[(t)^2-(t)^2]\sin(tx)}{[2(t)(t)]^2+[(t)^2-(t)^2]}\right]$$

$$= -\left[\frac{2(t)^2\cos(tx)}{4(t^2)^2}\right]$$

$$= -\left[\frac{\cos tx}{2t^2}\right] \qquad (\because \textbf{from eq}^n(2))$$

$$= -\left[\frac{\cos(k+1)x}{2(k+1)^2}\right]$$

6. Conclusion

In this paper we are able to derive the general formulae for finding P.I., By using mathematical induction method are given below,

- **For Cosine Formula:-**

$$\textbf{P.I.} = \left[\frac{2mn\sin nx - [n^2-m^2]\cos nx}{(2mn)^2 + [n^2-m^2]}\right]$$

- **For Sine Formula:-**

$$\textbf{P.I.} = -\left[\frac{2mn\sin nx + [n^2-m^2]\cos nx}{(2mn)^2 + [n^2-m^2]}\right]$$

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