A study on Hermitian, unitary and orthogonal operators

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Abstract:

A hermitian operator plays an important role in the theory of matrices. It includes Hermitian matrices and enjoys several of the same properties as Hermitian matrices. Indeed, while we proved that Hermitian matrices are unitary operators, we did not establish any converse; Hermitian operators are also unitarily operators. In this present paper, we have tried to establish the proper relation of Hermitian operators with others.

IndexTerms -

Matrices, Hermitian, Eigenvalues, Eigenvectors of Orthogonal and Unitary Operators, orthogonal operators.

I. INTRODUCTION

In this section, we study line operators on real inner product spaces that are called orthogonal operators and their complex corresponding item called unitary operators. There are operators however for which eigenvectors are orthogonal operators, and hence it is possible to have a basis that is simultaneously orthonormal and consist of eigenvectors. This chapter introduces some of these operators.

DEFINITION:

The Hermitian conjugate of a complex matrix A is the transpose of its complex conjugate $\bar{A}^T$

Example:

Hermitian conjugate of $\begin{bmatrix} 1+i & 3 & 3-i \\ 2-3i & 9i & 1+7i \\ 3 & 3-9i & 8+4i \end{bmatrix}$ is $\begin{bmatrix} 1-i & 2+3i & 3 \\ 3 & -9i & 3+9i \\ 3+i & 1-7i & 8-4i \end{bmatrix}$

DEFINITION:

A square complex matrix A is said to be Hermitian if it is equal to its Hermitian conjugate, A = $\bar{A}^T$

If L is a line operator on a complex inner product space V we can associate a complex matrix A with L so that mapping a vector v can be written as matrix multiplication

$$L(v) = Av$$

We use Hermitian matrices to define Hermitian operators.

DEFINITION:

A linear operator H on a complex inner product space is said to be Hermitian if its matrix is Hermitian.

H represents a Hermitian operator to distinguish that it is Hermitian. The matrix A of a Hermitian operator satisfies the equation A = $\bar{A}^T$ and for such an operator equation (u,Av) = $(\bar{A}^T u, v)$ implies that

$$(u, H(v)) = (H(u), v)$$

Hermitian operators are first of our operators for which eigenvectors are orthogonal. This is proved in the following theorem.

Theorem

Eigenvalues of a Hermitian operator on an interior product space are real and eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof

Let λ and v be an Eigen pair for a Hermitian operator H. Not out of necessity, but to simplify calculations, suppose that v has length one. The

$$\begin{aligned} \lambda &= \lambda(v,v) \\ &= (v,\lambda v) \\ &= \big(v,H(v)\big) \\ &= (H(v),v) \\ &= (\lambda v,v) \\ &= \bar{\lambda}(v,v) \\ &= \bar{\lambda} \end{aligned}$$

Hence λ is real now suppose v_1, v_2 are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2 then

$$\lambda_1(v_1, v_2) = (\lambda_1 v_1, v_2)$$
$$= (H(v_1), v_2)$$
$$= \left(v_1, H(v_2)\right)$$
$$= (v_1, \lambda_2 v_2)$$
$$\lambda_1(v_1, v_2) = \lambda_2(v_1, v_2)$$

Since $\lambda_1 \neq \lambda_2$ this implies that $(v_1, v_2) = 0$ and the eigenvectors are orthogonal

Definition:

An isomorphism T: V→V of interior product space. If $\|T(x)\| = \|x\|$ for all x ∈ V, we call T a unitary if F = C and an orthogonal operator if F = R.

Facts:

- Let u = u(n) = u(V) denote the set of all unitary operators on V.
- I ∈ u(V)
- U_1U_2 ∈ u(V) if U_1, U_2 ∈ u(V)
- U^{-1} ∈ u(V) if U ∈ u(V)
- So u (V) is a group under composition. It is a subgroup of the group of linear isomorphism of V. notationally
  U (V) ⊆ GL(V) ⊆ L(V, V)
- If V is finite dimensional then a linear operator T ∈ L(V, V) is unitary iff T preserves inner product

THEOREM:

Let F = R or F= C and let V be an interior product space over F. Let U∈ L(V, V) be a line operator. Then U is unitary if the adjoint U^* of U exists and $UU^* = U^*U = I$

Proof :

Suppose U is unitary. U^{-1} is an inverse of U. So for x,y ∈ V

We have (U (x),y) = (U(x),UU^{-1}(y)) = (x, U^{-1}(y))

So U^* exists and $U^* = U^{-1}$.

Conversely,

assume the adjoint U^*exists and $UU^* = U^*U = I$. we need to prove that U preserves the inner product. For x,y ∈ V

We have $\left(U(x), U(y)\right) = \left(x, U^*U(y)\right) = (x, y)$

Hence the proof.

THEOREM:

Let F =R or F = C and A be an $n \times n$ conditions. Let T: $F^n \to F^n$ be the line operator defined by T(x) = Ax. With usual the interior product on F^n, we have T is unitary if and only if $A^*A = I$

PROOF:

Suppose $A^*A = I$ then $A^*A = AA^* = I$ therefore

$$\left(T(x), T(y)\right) = (Ax, Ay)$$
$$= y^*A^*Ax$$
$$= y^*x = (x, y)$$

Conversely,

suppose T is unitary. Then $y^*A^*Ax = y^*x$ for all x,y $\in F^n$.

With appropriate choice of x,y we can show that $A^*A = I$. Hence the proof.

Conclusion:

In the theory of matrices, Hermitian matrices and its properties beers very large range of new results. The present subject matter related to the study of hermitian matrices is not very exhaustive. It is known that the normal matrices are perfectly conditioned with respect to the problem of finding their eigenvalues. We have tried to correlate and present a variety of problems of hermitian matrices.

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