RADIALLY SYMMETRIC POSITIVE SOLUTIONS FOR ELLIPTIC BOUNDARY VALUE PROBLEM IN R^{n}

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ABSTRACT

Symmetry properties of positive solutions for elliptic boundary value problems in R^{n} are considered. We employ the moving plane method based on maximum principle on unbounded domains to obtain the result on symmetry of solutions.

Key Words; Moving Plane Method, Maximum Principles, Elliptic Boundary value problems, Radial symmetry.

INTRODUCTION

In this paper we study the radial symmetry of positive solutions for elliptic boundary value problem in R^{n}. We consider the problem of the form

$$\Delta u + b_1 \frac{\partial u}{\partial x_1} + b_2 \frac{\partial u}{\partial x_2} + \dots\dots\dots + b_n \frac{\partial u}{\partial x_n} + f(|x|, u) = 0$$

$$u(x) \to 0 \;\; as \;\; |x| \to \infty \qquad when\; n \geq 2$$

We establish the symmetry result for the boundary value problem (1.1) in general case. Our arguments are based on the moving plane method. The device goes back to Alexandrov [8] and was first developed by J. Serrin [16], in the theory of partial differential equations, and later it was extended and generalized by Gidas, Ni and Nirenberg [5,6].The moving plane method has been further improved and simplified by Berestycki and Nirenberg [1,2] with the aid and Cong-ming Li [9]. Riechel [17] obtained symmetry results for semilinear elliptic boundary value problem in exterior domain.

Y Naito [10,11] obtained symmetry result for semilinear elliptic equations. Further Naito [12] studied the semilinear elliptic problem $\Delta u + f(|x|, u) = 0$ in R^{n}

In paper [13] author studied the symmetric solutions of nonlinear elliptic Neumann BVP

$$\Delta u = -1 \quad in\, \Omega \quad \text{with } u = 0$$

$$\frac{\partial u}{\partial \eta} = constant$$

Our proofs shows that the technique used by Berestycki and Nirenberg [1,2], Gidas, Ni and Nirenberg [5,6], Serrin [16], Cafarelli, Gidas and Spruck [3], Patil and Dhaigude [14] are useful for the study of symmetry of solutions of the elliptic boundary value problems.

In this paper we present an approach based on the maximum principle in unbounded domains together with the method of moving planes. In section 1 we state the main result about the symmetry of solutions and discuss the method of moving planes. Second section is devoted to the statements and proofs of some essential lemmas. Third section contains proof of the main result about symmetry of positive solutions of boundary value problem.

1. STATEMENT OF MAIN RESULT

In the boundary value problem

$$\Delta u + b_1 \frac{\partial u}{\partial x_1} + b_2 \frac{\partial u}{\partial x_2} + \dots\dots\dots + b_n \frac{\partial u}{\partial x_n} + f(|x|, u) = 0 \qquad (1.1)$$

$$u(x) \to 0 \;\; as \;\; |x| \to \infty \quad when\; n \geq 2, \quad b_1, b_2, \dots, b_n \in [0, \infty)$$

Assume that $b_1, b_2, \dots, b_n \in [0, \infty)$

f(r, u) is continuous and C^1 in $u \geq 0$ and f(r, u) is nonincreasing in r > 0 for each fixed $u \geq 0$.

Let $u \in C^2(R^n)$ be the solution of the equation (1.1) .

Define $U(r) = \sup\{u(x) : |x| \geq r\}$ and

$$\Phi(r) = \sup\left\{\frac{\partial f}{\partial u}(r, s): 0 \leq s \leq \{\sup u(x) |\, |x| \geq r\}\right\}$$

Assume that there exists a positive function w on $|x| \geq R_0$ for some $R_0 > 0$ satisfying

$$\Delta w + b_1 \frac{\partial w}{\partial x_1} + b_2 \frac{\partial w}{\partial x_2} + \dots\dots\dots + b_n \frac{\partial w}{\partial x_n} + \Phi(|x|) w \leq 0 \; in \; |x| \geq R_0 \qquad (1.2)$$

and

$$\lim_{|x| \to \infty} \frac{U(|x|)}{w(x)} = 0 \qquad (1.3)$$

Then u must be radially symmetric about some point $x_0 \in R^n$ and $u_r < 0\; for\; r > 0$.

Before proving the result we give the outline of the moving plane method.[4]

i. Consider the Euclidean space R^n for an example.
ii. Let u be the positive solution of a certain problem.
iii. If we want to show that the solution u is symmetric and monotone in the given direction then assume that direction as X_1-axis.
iv. For any real number λ, let $T_\lambda = \{x = (x_1, x_2, \dots, x_n) : x_1 = \lambda\}$.

    This plane is perpendicular to X_1 direction and it is the plane that we will move.

v. Let Σ_λ denote the region to the left of the plane. i.e. $\Sigma_\lambda = \{x =\in R^n : x_1 < \lambda\}$.
vi. Let x^λ be the reflection of the point x about the plane T_λ . i.e. $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$
vii. We compare the values of solution u at these two points x and x^λ. We want to show that u is symmetric about some plane $T_{\lambda 0}$. For this purpose we have to check that values of u at these points must be same.
viii. Let $V_\lambda(x) = u(x) - u(x^\lambda)$.
ix. In order to show that there exists some λ_0 such that $V_{\lambda 0}(x) = 0 \; for\; all\; x \in \Sigma_{\lambda 0}$
    We generally do this through the following two steps.

Step I : Prepare to move the plane.

Show that for λ is sufficiently negative, we have $V_\lambda(x) \geq 0 \;\; for\; all\; x \in \Sigma_\lambda$.Then we are able to start off from this neighborhood at $x_1 = -\infty$ and move the plane T_λ along the X_1- direction to the right as long as the inequality $V_\lambda(x) \geq 0 \;\; for\; all\; x \in \Sigma_\lambda$ holds.

Step II:Moving the plane

We continuously move this plane this way up to its limiting position. Define

$\lambda_0 = \sup\{\lambda : V_\lambda(x) \geq 0 \;\; for\ all\ x \in \Sigma_\lambda\}$.

We prove that u is symmetric about the plane $T_{\lambda 0}$ $i.e. V_{\lambda 0}(x) = 0 \;\; for\ all\ x \in \Sigma_{\lambda 0}$.

This is usually carried out by the method of contradiction.

We show that if $V_{\lambda 0}(x) \neq 0$ then there would exist $\lambda \geq 0$ such that $V_\lambda(x) \geq 0 \;\; for\ all\ x \in \Sigma_\lambda$ holds and this contradicts to the definition of λ_0.

Thus key to the moving plane method is to establish inequality$V_\lambda(x) \geq 0 \;\; for\ all\ x \in \Sigma_\lambda$. For this task in partial differential equations Maximum principles are powerful tools.

2. STATEMENTS AND PROOFS OF SOME ESSENTIAL LEMMAS:

Before going to prove main result we will state some useful results. We also state and prove some useful lemmas.

Theorem 2.1 MEAN VALUE THEOREM [15]: If u is harmonic in D..$u(\bar{x}, \bar{y})$is equal to its mean value taken over any circle in D with center at $(\bar{x}, \bar{y})$ and

$$u(\bar{x}, \bar{y}) = \frac{1}{2\pi R} \int_R uds$$

Theorem 2.2 STRONG MAXIMUM PRINCIPLE [15]: Suppose that $u \neq 0$ satisfies $L(u) \leq 0$in Ω and $u \geq 0$ on Ω. Furthermore that there exist a function $w > 0$ on $\Omega \cup \partial\Omega$ and $L(w) \leq 0 \;\; in\ \Omega$. If $\frac{u(x)}{w(x)} \to 0$. as $|x| \to \infty, x \in \Omega$, then $u > 0\ in\ \Omega$

Lemma 2.1 HOPF BOUNDARY LEMMA [7]: Suppose that Ω satisfies the interior sphere condition at $x_0 \in \partial\Omega$. Let L be strictly elliptic with $c \leq 0$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $L(u) \geq 0$ and

$$\max_{\bar{\Omega}} u(x) = u(x_0)$$

Then, either $u = u(x_0) in\ \Omega \;\; or \;\; \liminf_{t \to 0} \frac{u(x_0) - u(x_0 + t\nu)}{t} > 0$, possibly infinity, for every direction ν pointing in to an interior sphere. If $u \in C^1(\Omega) \cup \{0\}$, then $\frac{\partial u}{\partial \nu} < 0$.

Lemma 2.2 If $\lambda > 0 \;\; then \; |x^\lambda| \geq |x| \;\; for\ x \in \Sigma_\lambda$

Proof: Let $\lambda > 0$

$$|x^\lambda|^2 = (2\lambda - x_1)^2 + {x_2}^2 + \ldots + {x_n}^2$$

$$= 4\lambda^2 - 4\lambda x_1 + {x_1}^2 + {x_2}^2 + \ldots + {x_n}^2$$

$$= 4\lambda(\lambda - x_1) + {x_1}^2 + {x_2}^2 + \ldots + {x_n}^2$$

$$= 4\lambda(\lambda - x_1) + |x|^2$$

$$\therefore \; |x^\lambda|^2 - |x|^2 = 4\lambda(\lambda - x_1)$$

As $\lambda > x_1$ for $x \in \Sigma_\lambda$ we have $\lambda - x_1 > 0$

$\therefore \; |x^\lambda|^2 - |x|^2 \geq 0$

$$\therefore \ \left|x^{\lambda}\right|^{2} \ \geq \ |x|^{2}$$

$$\left|x^{\lambda}\right| \ \geq \ |x| \ \ for\ x \ \in \Sigma_{\lambda}$$

Lemma 2.3 Let > 0 , $b_i\ and\ \frac{\partial u}{\partial x_i}$ have same sign then V_λ satisfies

$$\Delta V_\lambda + b_1 \frac{\partial V_\lambda}{\partial x_1} + b_2 \frac{\partial V_\lambda}{\partial x_2} + \dots + b_n \frac{\partial V_\lambda}{\partial x_n} + C_\lambda(x) V_\lambda \leq 0 \qquad in \ \ \Sigma_\lambda$$

$$\because \ C_\lambda(x) \ = \ \int_0^1 \mathrm{f_u}\left(|\mathrm{x}|, \mathrm{u}\left(\mathrm{x}^\lambda\right) + \mathrm{t}\left(\mathrm{u(x)} - \mathrm{u}\left(\mathrm{x}^\lambda\right)\right)\right) \mathrm{dt}$$

Proof: Let $\Delta u + b_1 \frac{\partial u}{\partial x_1} + b_2 \frac{\partial u}{\partial x_2} + \dots + b_n \frac{\partial u}{\partial x_n} + f(|x|, u) = 0$ be a semilinear elliptic equation. Suppose $u(x) = u(x_1, x_2, x_3, \dots, x_n)$

$$\therefore u\left(x^\lambda\right) = u(2\lambda - x_1, x_2, \qquad x_3, \dots, x_n)$$

Differentiating we get $\frac{\partial u(x^\lambda)}{\partial x_1} = \frac{\partial u(x)}{\partial x_1}(-1), \frac{\partial u(x^\lambda)}{\partial x_2} = \frac{\partial u(x)}{\partial x_2}, \frac{\partial u(x^\lambda)}{\partial x_3} = \frac{\partial u(x)}{\partial x_3}, \dots, \frac{\partial u(x^\lambda)}{\partial x_n} = \frac{\partial u(x)}{\partial x_n}$

Differentiating second time

$\frac{\partial^2 u(x^\lambda)}{\partial {x_1}^2} = \frac{\partial^2 u(x)}{\partial {x_1}^2}, \frac{\partial^2 u(x^\lambda)}{\partial {x_2}^2} = \frac{\partial^2 u(x)}{\partial {x_2}^2}, \frac{\partial^2 u(x^\lambda)}{\partial {x_3}^2} = \frac{\partial^2 u(x)}{\partial {x_3}^2}, \dots, \frac{\partial^2 u(x^\lambda)}{\partial {x_n}^2} = \frac{\partial^2 u(x)}{\partial {x_n}^2}$

Therefore we have following two equations

$$\Delta u(x) + b_1 \frac{\partial u(x)}{\partial x_1} + b_2 \frac{\partial u(x)}{\partial x_2} + \dots + b_n \frac{\partial u(x)}{\partial x_n} + f(|x|, u(x)) = 0$$

$$\Delta u(x^\lambda) + b_1 \frac{\partial u(x^\lambda)}{\partial x_1} + b_2 \frac{\partial u(x^\lambda)}{\partial x_2} + \dots + b_n \frac{\partial u(x^\lambda)}{\partial x_n} + f\left(\left|x^\lambda\right|, u(x^\lambda)\right) = 0$$

Subtracting we get,

$$\begin{aligned} 0 = &\left[\Delta u(x) + b_1 \frac{\partial u(x)}{\partial x_1} + b_2 \frac{\partial u(x)}{\partial x_2} + \dots + b_n \frac{\partial u(x)}{\partial x_n} + f(|x|, u(x))\right] \\ &- \left[\Delta u(x^\lambda) + b_1 \frac{\partial u(x^\lambda)}{\partial x_1} + b_2 \frac{\partial u(x^\lambda)}{\partial x_2} + \dots + b_n \frac{\partial u(x^\lambda)}{\partial x_n} + f\left(\left|x^\lambda\right|, u(x^\lambda)\right)\right] \end{aligned}$$

$$\begin{aligned} = &\Delta u(x) - \Delta u\left(x^\lambda\right) + \left(b_1 \frac{\partial u(x)}{\partial x_1} + b_2 \frac{\partial u(x)}{\partial x_2} + \dots + b_n \frac{\partial u(x)}{\partial x_n}\right) - \left(b_1 \frac{\partial u(x^\lambda)}{\partial x_1} + b_2 \frac{\partial u(x^\lambda)}{\partial x_2} + \dots + b_n \frac{\partial u(x^\lambda)}{\partial x_n}\right) \\ &+ f(|x|, u(x)) - f\left(\left|x^\lambda\right|, u(x^\lambda)\right) \end{aligned}$$

$$\begin{aligned} = &\Delta\left(u(x) - u\left(x^\lambda\right)\right) + b_1 \left(\frac{\partial u(x)}{\partial x_1} - \frac{\partial u(x^\lambda)}{\partial x_1}\right) + b_2 \left(\frac{\partial u(x)}{\partial x_2} - \frac{\partial u(x^\lambda)}{\partial x_2}\right) + b_3 \left(\frac{\partial u(x)}{\partial x_3} - \frac{\partial u(x^\lambda)}{\partial x_3}\right) + \dots + b_n \left(\frac{\partial u(x)}{\partial x_n} - \frac{\partial u(x^\lambda)}{\partial x_n}\right) \\ &+ \ + f(|x|, u(x)) - f\left(\left|x^\lambda\right|, u(x^\lambda)\right) \end{aligned}$$

$$= \Delta\left(V_\lambda(x)\right) + b_1 \frac{\partial (V_\lambda(x))}{\partial x_1} + b_2 \frac{\partial (V_\lambda(x))}{\partial x_2} + b_3 \frac{\partial (V_\lambda(x))}{\partial x_3} + \dots + b_n \frac{\partial (V_\lambda(x))}{\partial x_n} + f(|x|, u(x)) - f\left(\left|x^\lambda\right|, u(x^\lambda)\right)$$

$$\geq \Delta\left(V_\lambda(x)\right) + \left(b_1 \frac{\partial}{\partial x_1} + b_2 \frac{\partial}{\partial x_2} + b_3 \frac{\partial}{\partial x_3} + \dots + b_n \frac{\partial}{\partial x_n}\right)(V_\lambda(x)) + f(|x|, u(x)) - f\left(|x|, u(x^\lambda)\right)$$

$$\geq \Delta\big(V_\lambda(x)\big) + \left(b_1\frac{\partial}{\partial x_1} + b_2\frac{\partial}{\partial x_2} + b_3\frac{\partial}{\partial x_3} + \ldots + b_n\frac{\partial}{\partial x_n}\right)\big(V_\lambda(x)\big) + \frac{f\big(|x|, u(x)\big) - f\left(|x|, u(x^\lambda)\right)}{u(x) - u(x^\lambda)} \times V_\lambda(x)$$

$$\geq \Delta\big(V_\lambda(x)\big) + \left(b_1\frac{\partial}{\partial x_1} + b_2\frac{\partial}{\partial x_2} + b_3\frac{\partial}{\partial x_3} + \ldots + b_n\frac{\partial}{\partial x_n}\right)\big(V_\lambda(x)\big) + C_\lambda(x)V_\lambda(x)$$

$$where\ C_\lambda(x) = \frac{f\big(|x|, u(x)\big) - f\left(|x|, u(x^\lambda)\right)}{u(x) - u(x^\lambda)}$$

$$i.e.\ C_\lambda(x) = \int_0^1 f_u\left(|x|, u(x^\lambda) + t\left(u(x) - u(x^\lambda)\right)\right) dt$$

Put $B_0 = \{x \in R^n : |x| < R_0\}$ Where R_0 is the constant defined in the statement of the theorem.

Define Λ as $\Lambda = \{x \in R^n : V_\lambda(x) \geq 0\ in\ \Sigma_\lambda\}$.

Lemma2.4 Let $\lambda > 0$. If $V_\lambda > 0\ \ on\ \Sigma_\lambda \cap \overline{B_0}$. Then $\lambda \in \Lambda$.

Proof: Let $\lambda > 0$. If $V_\lambda > 0\ \ on\ \Sigma_\lambda \cap \overline{B_0}$. From lemma 2.3 and assumption

$$\Delta V_\lambda + \left(b_1\frac{\partial}{\partial x_1} + b_2\frac{\partial}{\partial x_2} + b_3\frac{\partial}{\partial x_3} + \ldots + b_n\frac{\partial}{\partial x_n}\right)v_\lambda + C_\lambda(x)V_\lambda(x) \leq 0\ in\ \Sigma_\lambda \setminus \overline{B_0}$$

$$V_\lambda \geq 0\ \ on\ \partial\,(\Sigma_\lambda \setminus \overline{B_0})$$

Since U(R) is non-increasing we have

$$0 \leq u(x^\lambda) + t\left(u(x) - u(x^\lambda)\right) \leq U(|x|)\ for\ 0 \leq t \leq 1$$

$$\therefore\ C_\lambda(x) = \int_0^1 f_u\left(u(x^\lambda) + t\left(u(x) - u(x^\lambda)\right)\right) dt$$

$$\leq \int_0^1 f_u(U(|x|))\, dt$$

$$= \phi(|x|)\ in\ \Sigma_\lambda$$

$$\because\ \phi(|x|) = \sup\{f_u(r, s) |\ 0 \leq s \leq U(r)\}$$

From $\Delta w + \left(b_1\frac{\partial}{\partial x_1} + b_2\frac{\partial}{\partial x_2} + b_3\frac{\partial}{\partial x_3} + \ldots + b_n\frac{\partial}{\partial x_n}\right)w + \phi(|x|)\, w \leq 0\ in\ |x| \geq R_0$

and $\lim\frac{U(|x|)}{w(x)} = 0$

The positive function w satisfies

$$\Delta w + \left(b_1\frac{\partial}{\partial x_1} + b_2\frac{\partial}{\partial x_2} + b_3\frac{\partial}{\partial x_3} + \ldots + b_n\frac{\partial}{\partial x_n}\right)w + C_\lambda(x)\, w \leq 0\ in\ \Sigma_\lambda \setminus \overline{B_0}$$

and $\frac{V_\lambda(x)}{w(x)} \leq \frac{u(|x|)}{w(x)} \to 0 \qquad x \in \Sigma_\lambda \setminus \overline{B_0}\ , |x| \to \infty$

Hence by maximum principle we have

$$V_\lambda > 0 \quad in\ \Sigma_\lambda \setminus \overline{B_0}$$

$$\therefore V_\lambda > 0 \quad in\ \Sigma_\lambda$$

By assumption

$\lambda \in \Lambda$.

Lemma 2.5 Let $\lambda > 0$. If $\lambda \notin \Lambda$ then $\frac{\partial u}{\partial x_1} < 0$ on T_λ, then there exist $x_0 \in \Sigma_\lambda \cap \overline{B_0}$ such that $V_\lambda(x_0) \leq 0$

Lemma 2.6 Let $\lambda \in \Lambda$ then $\frac{\partial u}{\partial x_1} < 0$ on T_λ.

Proof: Let $\lambda \in \Lambda$ Hence $\lambda > 0$

By lemma 2.3

$$\Delta V_\lambda + \left(b_1 \frac{\partial}{\partial x_1} + b_2 \frac{\partial}{\partial x_2} + b_3 \frac{\partial}{\partial x_3} + \ldots + b_n \frac{\partial}{\partial x_n}\right) v_\lambda + C_\lambda(x) V_\lambda(x) \leq 0 \; in \; \Sigma_\lambda$$

$$V_\lambda > 0 \quad in \; \Sigma_\lambda$$

Therefore on T_λ , $\therefore V_\lambda(x) = 0$

ByHopf boundary lemma

$$\frac{\partial V_\lambda}{\partial x} < 0 \quad on \; T_\lambda$$

$$\therefore \frac{\partial u}{\partial x_1} = \frac{1}{2} \frac{\partial V_\lambda}{\partial x_1} < 0 \; on \; T_\lambda$$

2. PROOF OF MAIN RESULT

Since u(x0 is positive and

$$\lim_{|x| \to \infty} u(x) = 0$$

then there exist $R_1 > R_0$ such that

$$\max\{u(x) : |x| > R_1\} < \min\{u(x) : |x| \leq R_0\} \qquad (2.1)$$

where R_0 is constant as defined in the statement of the theorem. We shall prove the theorem in following three steps.

Step I : To prove $[R_1, \infty) \subset \Lambda$

Let $\lambda \in [R_1, \infty)$

$$\therefore \lambda \geq R_1$$

Note that $\overline{B_0} \subset \Sigma_\lambda$

Also $V_\lambda(x) = u(x) - u(x^\lambda)$

From 2.1 $V_\lambda(x) > 0 \; in \; \overline{B_0}$

By lemma 2.4 $\lambda \in \Lambda$

$$[R_1, \infty) \subset \Lambda$$

Step II: Toprove: Let $\lambda_0 \in \Lambda$ then there exist $\epsilon > 0$ such that $(\lambda_{0,} - \epsilon, \lambda_0) \subset \Lambda$.

Assume to the contrary that there exist *an increasing sequence* $\{\lambda_i\}$i = 1,2,3 …..

such that $\lambda_i \notin \Lambda$ and $\lambda_i \to \lambda_0$ as $i \to \infty$. By lemma 2.5 we have sequence $\{x_i\}$i = 1,2,3 …..such that $x_i \in \Sigma_{\lambda_i \cap \overline{B_0}}$ and $V_{\lambda i}(x_i) \leq 0$. A subsequence which we call again $\{x_i\}$ $i = 1,2,3, \ldots$ converges to some point$x_0 \in \overline{\Sigma_{\lambda 0}} \cap \overline{B_0}$.

Then $V_{\lambda 0}(x_0) \leq 0$.

Since $V_{\lambda 0} > 0$in $\Sigma_{\lambda 0}$. We have $x_0 \in T_{\lambda 0}$.

By mean value theorem there exist a point y_i satisfying$\left(\frac{\partial u}{\partial x_i}\right)_{yi} \geq 0$ on straight segment joining x_i *to* ${x_i}^{\lambda_i}$ for each i = 1,2,3…. Since $y_i \to x_0$ as $i \to \infty$ we have $\frac{\partial u}{\partial x_1}(x_0) \geq 0$.

On the other hand $x_0 \in T_{\lambda 0}$, we have $\frac{\partial u}{\partial x_1}(x_0) < 0$ by lemma 2.6. This is a contradiction. Hence our assumption is wrong.

$\therefore (\lambda_{0,} - \epsilon, \lambda_0) \subset \Lambda$

Step III :To prove either statement (A) or statement (B) holds.

(A) $u(x) = u(x^{\lambda 1})$for some $\lambda_1 > 0$ and $\frac{\partial u}{\partial x_1} < 0 \; on \; T_\lambda$ for $\lambda > \lambda_1$.

(B) $u(x) = u(x^0)$in $\Sigma_0$and $\frac{\partial u}{\partial x_1} < 0 \ \ on \ T_\lambda$ for $\lambda > 0$

Proof of step III : Let $\lambda_1 = \inf\{\lambda > 0 : (\lambda, \infty) \subset \Lambda\}$
We distinguish it in to two cases
Case (1) $\lambda_1 > 0$
Let $V_{\lambda 1}(x) = u(x) - u(x^{\lambda 1})$
Since u is continuous $V_{\lambda 1}(x) \geq 0$in $\Sigma_{\lambda 1}$. From lemma 2.3. we have

$$\Delta V_\lambda(x) + \left(b_1 \frac{\partial}{\partial x_1} + b_2 \frac{\partial}{\partial x_2} + b_3 \frac{\partial}{\partial x_3} + \ \ldots + \ b_n \frac{\partial}{\partial x_n}\right) v_\lambda(x) + \ C_\lambda(x) V_\lambda(x) \leq 0 \ in \ \Sigma_{\lambda 1}$$

Hence by strong maximum principle we have that either $V_{\lambda 1} > 0 \ \ in \ \Sigma_{\lambda 1}$ or $V_{\lambda 1} = 0 \ \ in \ \Sigma_{\lambda 1}$

Assume that $V_{\lambda 1} > 0 \ \ in \ \Sigma_{\lambda 1}$then $\lambda_1 \ \in \Lambda$. by lemma 2.4.

From step II there exist $\epsilon > 0$ such that $(\lambda_1 - \epsilon, \lambda_1) \subset \Lambda$. This contradicts to the definition of λ_1.

$$\therefore V_{\lambda 1} = 0 \ \ in \ \Sigma_{\lambda 1}$$

$$\therefore u(x) = u\left(x^{\lambda_1}\right) \quad for \ \lambda_1 > 0$$

Since $(\lambda_1, \infty) \subset \Lambda$.We have $\frac{\partial u}{\partial x_1} < 0 \ \ on \ T_\lambda$ for $\lambda > \lambda_1$ by lemma 2.6.

Thus statement (A) holds.

Case II: Let $\lambda_1 = 0$Since u is continuous$u(x) \geq u(x^0) \ \ in \ \Sigma_0$ by lemma 2.6 $\frac{\partial u}{\partial x_1} < 0 \ \ on \ T_\lambda$ for$\lambda > \ 0$. Thus statement (B) holds. If (B) occurs in step III we can repeat the previous steps I II and III for negative X_1 direction about some planr$x_1 = \ \lambda_1 < 0$ or $u(x) \leq u(x^0) \quad in \ \Sigma_0$ Therefore $u(x) + u(x^0) \ \ in \ \Sigma_0$

Thus u must be radially symmetric in X_1 direction about some plane and strictly decreasing away from the plane. As the given equationis invariant under rotation we may take any direction as X_1 directionand conclude that u is symmetric in every direction about some plane. Therefore u is radially symmetric about some point $x_0 \ \in \ R^n$and$u_r < o \ \ for \ r > 0$.

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