Some Fixed Point Theorems in Dislocated Quasi b Metric Spaces

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Abstract: In this paper, we are giving fixed point results in dislocated quasi b metric space with some new contractive conditions.
Keyword: Metric spaces, Cauchy sequences, complete metric spaces, contraction mappings.

I. Introduction

Fixed point theory is the most dynamic subject of Non-linear sciences. Because of its feasibility of applications in various disciplines of sciences and other fields, many researchers have given their contribution, and several research articles are published in this area. The most crucial result which attracted most of the researchers is Banach Contraction Principal [1], given by Banach in his thesis in 1922, which asserts that 'Every contraction mapping on complete metric space has a unique fixed point.' This theorem provides existence and uniqueness of the solution and also provides a systematic way to find a solution and the existence theorems can be expressed in the form of fixed point principles. Therefore, it becomes an active area of research in nonlinear analysis with vast applications. After this theorem Banach contraction principle is presented in various forms by various researchers either by using different contractive conditions on mappings or different generalizations of the topologies of the metric spaces and exciting results are obtained. Some generalizations of mappings are Kannan contraction, Ciric contraction T-Kannan contraction, T-Banach contraction, weakly contraction, cyclic contraction, d-cyclic φ-contraction, Chatterjee type contraction, α-φ contractive mappings, etc. Some generalizations of metric spaces are partial metric space, cone metric space, b-metric space, G-Metric space, dislocated metric space, quasi-metric space, b-metric space, dislocated quasi-metric space, dislocated quasi b-metric space, modular metric space, etc.

1.1 Definition: Let X be a nonempty set, suppose that the mapping $d: X \times X \to [0, \infty)$ satisfies the following conditions:

i. $d(x, x) = 0$ for all $x \epsilon X$;

ii. $d(x, y) = d(y, x) = 0$ implies $x = y$ for all $x, y \epsilon X$

iii. $d(x, y) = d(y, x)$ for all $x, y \epsilon X$;

iv. $d(x, y) \leq d(x, z) + d(z, y)$ For all $x, y, z \epsilon X$.

Then d is called a metric on X and (X, d) is called as Metric space. Further, if d satisfies conditions (i), (ii) and (iii), then d is called a quasi-metric on X. If d satisfies conditions (ii), (iii) and (iv), then d is called a dislocated metric on X.

In 2000, Hitzler and Seda [3] excluded the condition of self distance equal to zero in the hypothesis of metric spaces and introduced the concept of dislocated metric spaces, where the self distance for any point in space need not be zero.

As a generalization of metric spaces, the concept of quasi-metric spaces was introduced by Wilson [3] by dropping the symmetric property in dislocated metric space, then F. M. Zeyada *et. al.*[4], put some definitions and strengthen the literature of generalization of metric spaces. In their study, they used the concept of dislocated metric space due to Hitzler and Seda [2] to establish the idea of complete dislocated quasi-metric space and proved new fixed point theorem in dislocated quasi-metric space. Next, to Hitzler and Seda [2], Zeyada *et al.* [4] gave another generalization of metric space as dislocated quasi-metric space by using the concept of dislocated metric space and further established a fixed point theorem in complete dislocated spaces. Bakhtin[5] familiarized the concept of b-metric space by relaxing the triangle inequality and further Czerwik[6] in his study of 'contraction mappings in b-metric spaces,' made more popular and gave new way, and so fixed point theory developed in the new class of b-metric spaces which is larger than that of class of metric spaces.

1.2 Definition: [6] Let X be a nonempty set. Suppose that mapping $b: X \times X \to [0, \infty)$ such that the constant $s \geq 1$satisfies the following conditions:

i. $b(x, y) = b(y, x) = 0 \Leftrightarrow x = y$ for all $x, y \epsilon X$

ii. $b(x, y) = b(y, x)$ for all $x, y \epsilon X$;

iii. $b(x, y) \leq s[b(x, z) + b(z, y)]$ For all $x, y, z \epsilon X$.

Then pair (X, b) is called a b-metric space.

1.3 Remark: Every metric space is b-metric space but not conversely.

The generalization of metric space as dislocated quasi-b-metric space was introduced by Chakkrid Klin-eam and Cholatis Suanoom[8] and also given the existence of fixed point theorems for dqb-metric spaces.

1.4 Definition: [8] Let X be a nonempty set. Suppose that the mapping $b: X \times X \to [0, \infty)$ such that the constant $s \geq 1$ satisfies the following conditions:

i. $d(x, y) = d(y, x) = 0$ implies $x = y$ for all $x, y \epsilon X$

ii. $d(x, y) \leq s[d(x, z) + d(z, y)]$ For all $x, y, z \epsilon X$.

Then pair (X, d) is called a dislocated, quasi b-metric space(or simply dqb-metric space). The number s is called the coefficient of(X, d).

Remark: The b-metric spaces, quasi-b-metric spaces are dislocated quasi-b-metric spaces, but the converse is not true.

C. T. Aage and J. N. Salunke [9,10] derived some results in dislocated and dislocated quasi-metric spaces and proved contraction theorem on dq-metric spaces for continuous mapping, Jha and Panthi [13], proved some contraction theorems in dislocated metric space, D. Panthi *et. al.* [12], MU Rahman *et al.* [14] showed some results on contraction principal in dislocated-quasi metric space. Mujeeb Ur Rahman and Muhammad Sarwar [15] has given some remarks in d-metric space and dq-metric spaces. In 2010, A. Isufati[11] derived some fixed point theorems for continuous contractive conditions in dislocated quasi-metric space. In 2016, Rahman and Sarwar [16] proved Banach's contraction principle, Kannan and Chatterjee type fixed point results for self-mapping in dislocated quasi b-metric space. Aage and Golhare [17] used different kinds of mapping and contractions like weakly compatible mappings, Banach contraction mapping, Kannan contraction mapping in dislocated quasi b-metric spaces and developed some common fixed point theorems in these spaces. Also, they proved the fixed points for α-admissible mappings in dislocated quasi b-metric spaces.

In this paper, we establish a new fixed point theorem in dislocated quasi b metric space using some new contractive conditions.

We require some definitions in dislocated quasi metric space.

1.5 Definition: [8] A sequence $\{x_n\}$ in a dqb-metric space (X, d) dislocated quasi-b-converges (or dqb-converges) to $x \epsilon X$ If $\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n)$

and x is called a dqb-limit of $\{x_n\}$, this can be written as $x_n \to x$.

1.6 Definition: [8] A sequence $\{x_n\}$ in a dqb-metric space (X, d) is called Cauchy if $\lim_{n,m\to\infty} d(x_n, x_m) = 0 = \lim_{n,m\to\infty} d(x_m, x_n)$

1.7 Definition: [8] A dqb-metric space (X, d) is complete if every Cauchy sequence in it is dqb-convergent in X.

II. Main Result

2.1 Theorem: Let (X, d) be a complete dqb-metric space with coefficient s. Suppose that the mapping $T: X \to X$ satisfies the following conditions:

$$d(Tx, Ty) \le \left(\frac{d(x, Tx) + d(y, Ty)}{d(x, Tx) + d(y, Ty) + k}\right) d(x, y)$$

For all $x, y \epsilon X$, where $k \ge 1$, then

i. T has a unique fixed point in X.

ii. $T^n x^*$ to a fixed point, for all $x^* \epsilon X$.

Proof:

(i.)Let $x_0 \epsilon X$ is arbitrary point in X we define a sequence $\{x_n\}$ in X by denoting $x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0, \ldots x_{n+1} = Tx_n = T^n x_0$.

Consider, $d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})$

$$\le \left(\frac{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})}{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1}) + k}\right) d(x_n, x_{n-1})$$

$$\le \left(\frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + k}\right) d(x_n, x_{n-1})$$

We take, $\beta_n = \left(\frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + k}\right)$

We have, $d(x_{n+1}, x_n) \le \beta_n d(x_n, x_{n-1})$

$$\le \beta_n \beta_{n-1} d(x_{n-1}, x_{n-2})$$

$$\le (\beta_n \beta_{n-1} \ldots \beta_1) d(x_1, x_0)$$

We observe here that, $\{\beta_n\}$ is the non-increasing sequence, with positive terms. Therefore $\beta_1 \beta_2 \ldots \beta_n \le \beta_1^n$ and also $\beta_1^n \to 0$ as $n \to \infty$.

It follows that,

$\lim_{n\to\infty} (\beta_n \beta_{n-1} \ldots \beta_1) = 0.$

Thus we get that, $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$

Now, for all $m, n \epsilon N$ and $m > n$

We have,

$$d(x_m, x_n) \le s^{m-n} d(x_m, x_{m-1}) + s^{m-n-1} d(x_{m-1}, x_{m-2}) + \cdots + s d(x_{n+1}, x_n)$$
$$d(x_m, x_n) \le s^{m-n} (\beta_{m-1} \beta_{m-2} \ldots \beta_1) d(x_1, x_0) + s^{m-n-1} (\beta_{m-2} \beta_{m-3} \ldots \beta_1)(d(x_1, x_0) + \cdots + s(\beta_n \beta_{n-1} \ldots \beta_1) d(x_1, x_0)$$
$$d(x_m, x_n) \le s^{m-n} \beta_1^{m-1} d(x_1, x_0) + s^{m-n-1} \beta_1^{m-2} d(x_1, x_0) + \cdots + s^2 \beta_1^{n+1} d(x_1, x_0) + s \beta_1^n d(x_1, x_0)$$

$$d(x_m, x_n) \le (s^{m-n} \beta_1^{m-1} + s^{m-n-1} \beta_1^{m-2} + \cdots + s^2 \beta_1^{n+1} + s \beta_1^n) d(x_1, x_0)$$

Take $n \to \infty$ we get, $d(x_m, x_n) \to 0$

Similarly, by using triangle inequality

We have

$$d(x_n, x_m) \le (s \beta_1^n + s^2 \beta_1^{n+1} + \cdots + s^{m-n-1} \beta_1^{m-2} + s^{m-n} \beta_1^{m-1}) d(x_0, x_1)$$

Take $n \to \infty$ we get, $d(x_n, x_m) \to 0$

Thus, we have $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete dqb-metric space, $\{x_n\}$ must converge to some point $x^* \epsilon X$, such that $x_n \to x^*$ as $n \to \infty$

$$d(Tx^*, x^*) \le s d(Tx^*, Tx_n) + s d(Tx_n, x^*)$$

$$\le s \left(\frac{d(x^*, Tx^*) + d(x_n, Tx_n)}{d(x^*, Tx^*) + d(x_n, Tx_n) + k}\right) d(x^*, x_n) + s d(x_{n+1}, x^*)$$

$$\le s \left(\frac{d(x^*, Tx^*) + d(x_n, x_{n+1})}{d(x^*, Tx^*) + d(x_n, x_{n+1}) + k}\right) d(x^*, x_n) + s d(x_{n+1}, x^*)$$

$d(Tx^*, x^*) \le 0$ as $n \to \infty$

Therefore $d(Tx^*, x^*) = 0$

Similarly we can prove that, $d(x^*, Tx^*) = 0$ and thus we have $Tx^* = x^*$

Uniqueness,

Suppose that x^* and z^* are two fixed points of T.

Consider, $d(x^*,y^*) = d(Tx^*,Ty^*)$

$$\leq \left(\frac{d(x^*,Tx^*)+d(y^*,Ty^*)}{d(x^*,Tx^*)+d(y^*,Ty^*)+k}\right)d(x^*,y^*)$$

Thus, $d(x^*,y^*) \leq 0 \Longrightarrow d(x^*,y^*) = 0 \Longrightarrow x^* = y^*$

Hence, T unique fixed point x^*.

To prove (ii)

Consider,

$d(T^n x^*,x^*) = d(T^{n-1}(Tx^*),x^*) = d(T^{n-1}x^*,x^*) = d(T^{n-2}(Tx^*),x^*) = \cdots = d(Tx^*,x^*) = 0$

Thus, $d(T^n x^*,x^*) = 0 \Rightarrow T^n x^* = x^*$

Proving that, $T^n x^*$ converges to a fixed point, for all $x^* \epsilon X$.

2.2 Corollary: Let (X,d) be a complete dqb-metric space with coefficient s. Suppose that the mapping $T\colon X \to X$satisfies the following conditions:

$$d(Tx,Ty) \leq \left(\frac{d(x,Tx)+d(y,Ty)}{d(x,Tx)+d(y,Ty)+1}\right)d(x,y)$$

For all $x,y\epsilon X$ then

i. T has unique fixed point in X.

ii. $T^n x^*$ to a fixed point, for all $x^*\epsilon X$.

Proof: The proof of the corollary immediately follows from the proof of the above theorem by putting k=1.

2.3 Theorem: Let (X,d) be a complete dislocated-quasi-b-metric space with coefficient s and let T be a mapping from X into itself. Suppose that T satisfies the following conditions:

$$d(Tx,Ty) \leq \left(\frac{d(y,Ty)}{d(x,Tx)+d(y,Ty)+k}\right)d(x,y)$$

For all $x,y\,\epsilon X$, where $k \geq 1$, then

(i) T has a unique fixed point in X.

(ii) $T^n x^*$ Converges to a fixed point, for all $x^*\epsilon\, X$.

Proof: (i)Let $x_0$be arbitrary point in X and choose a sequence $\{x_0\}$ such that, $x_{n+1} = Tx_n$.

Consider,

$d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1})$

$$\leq \left(\frac{d(x_{n-1},Tx_{n-1})}{d(x_n,Tx_n)+d(x_{n-1},Tx_{n-1})+k}\right)d(x_n,x_{n-1})$$

$$\leq \left(\frac{d(x_{n-1},x_n)}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+k}\right)d(x_n,x_{n-1})$$

We take, $\beta_n = \left(\frac{d(x_{n-1},x_n)}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+k}\right)$

We get that, $d(x_{n+1},x_n) \leq \beta_n d(x_n,x_{n-1}) \leq \beta_n\beta_{n-1}d(x_{n-1},x_{n-2})$

$$d(x_{n+1},x_n) \leq \beta_n\beta_{n-1}\ldots\beta_1 d(x_1,x_0).$$

Observe that, $\{\beta_n\}$ is a non-increasing sequence, with positive terms.

So, $\beta_n\beta_{n-1}\ldots\beta_1 \leq \beta_1^n$ and $\beta_1^n \to 0$ as $n \to \infty$.

It follows that, $\lim_{n\to\infty}\beta_n\beta_{n-1}\ldots\beta_1 = 0$

Thus we get, $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$

Now for all $m,n\,\epsilon\,\mathbb{N}$ and $m > n$

We have,

$$d(x_m,x_n) \leq s^{m-n}d(x_m,x_{m-1}) + s^{m-n-1}d(x_{m-1},x_{m-2}) + \cdots s^2 d(x_{n+2},x_{n+1}) + sd(x_{n+1},x_n)$$

$d(x_m,x_n) \leq s^{m-n}\beta_{m-1}\beta_{m-2}\ldots\beta_1 d(x_1,x_0) + s^{m-n-1}\beta_{m-2}\beta_{m-3}\ldots\beta_1 d(x_1,x_0) + \cdots + s^2\beta_{n+1}\beta_n\ldots\beta_1 d(x_1,x_0) + s\beta_n\beta_{n-1}\ldots\beta_1 d(x_1,x_0)$

$d(x_m,x_n) \leq s^{m-n}\beta_1^{m-1}d(x_1,x_0) + s^{m-n-1}\beta_1^{m-2}d(x_1,x_0) + \cdots + s^2\beta_1^{n+1}d(x_1,x_0) + s\beta_1^n d(x_1,x_0)$

$d(x_m,x_n) \leq (s^{m-n}\beta_1^{m-1} + s^{m-n-1}\beta_1^{m-2} + \cdots + s^2\beta_1^{n+1} + s\beta_1^n)d(x_1,x_0)$

Take $n \to \infty$ we get, $d(x_m,x_n) \to 0$

Similarly we can prove, $d(x_n,x_m) \to 0$

Thus $\{x_n\}$ is a Cauchy sequence.

Now, since the (X,d) is a complete dqb-metric space, the sequence $\{x_n\}$ must converge to some point x^* in (X,d), such that $x_n \to x^*$ as $n \to \infty$.

$$\lim_{n\to\infty} d(x_n,x^*) = 0 = \lim_{n\to\infty} d(x^*,x_n)$$

$$d(Tx^*,x^*) \leq sd(Tx^*,Tx_n) + sd(Tx_n,x^*)$$

$$\leq s\left(\frac{d(x_n,Tx_n)}{d(x^*,Tx^*)+d(x_n,Tx_n)+k}\right)d(x_n,x^*) + sd(Tx_n,x^*)$$

$$\leq s\left(\frac{d(x_n,x_{n+1})}{d(x^*,Tx^*)+d(x_n,x_{n+1})+k}\right)d(x_n,x^*) + sd(x_{n+1},x^*)$$

As $n \to \infty$, $d(Tx^*,x^*) \leq 0$ this implies that $d(Tx^*,x^*) = 0$

Similarly we can prove $d(x^*,Tx^*) = 0$

Thus, $Tx^* = x^*$ giving that x^* is a fixed point of T.

Uniqueness:

Consider two fix point x* and y* of T. Therefore,$Tx^* = x^*$ and $Ty^* = y^*$.

Consider,

$$d(x^*,y^*) = d(Tx^*,Ty^*) \le \left(\frac{d(y^*,Ty^*)}{d(x^*,Tx^*)+d(y^*,Ty^*)+k}\right)d(x^*,y^*)$$
$$d(x^*,y^*) \le 0 \Longrightarrow d(x^*,y^*) = 0 \Longrightarrow x^* = y^*$$

Hence x^* is an unique fixed point of T.

(ii)Now,

Consider,$d(T^n x^*,x^*) = d(T^{n-1}(Tx^*),x^*) = d(T^{n-1}x^*,x^*) = \cdots = d(Tx^*,x^*) = 0$

$i.e$ $d(T^n x^*,x^*) = 0$,

Similarly we can prove that $d(x^*,T^n x^*) = 0$

Hence $\{T^n x^*\}$ converges to a fixed point, for all $x^* \epsilon X$.

2.4 Corollary: Let (X,d)be a complete dislocated quasi-metric space with coefficient s and let T be a mapping from X into itself.

Suppose that Tsatisfies the following condition:

$$d(Tx,Ty) \le \left(\frac{d(y,Ty)}{d(x,Tx)+d(y,Ty)+1}\right)d(x,y)$$

For all $x,y \epsilon X$, then

(i) T has unique fixed point in X.

(ii) $\{T^n x^*\}$ Converges to a fixed point, for $x^* \epsilon X$.

Proof: The proof of the corollary immediately follows from the proof of the above theorem by putting k=1.

2.5 Theorem: Let (X,d) be a complete dqb-metric space with coefficient s. Suppose the mapping $T: X \to X$ satisfies the following conditions:

$$d(Tx,Ty) \le \left(\frac{d(y,Ty)+d(y,Tx)}{d(y,Ty)\cdot d(y,Tx)+k}\right)d(x,y)$$

For all $x,y \epsilon X$, where $k \ge 1$then

(i) T has a unique fixed point in X.

(ii) $\{T^n x*\}$ Converges to a fixed point, for all $x^* \epsilon X$.

Proof: (i) Let $x_0 \epsilon X$ be arbitrary point in X and choose a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\le \left(\frac{d(x_n,Tx_n)+d(x_n,Tx_{n-1})}{d(x_n,Tx_n)\cdot d(x_n,Tx_{n-1})+k}\right)d(x_{n-1},x_n) \\ &\le \left(\frac{d(x_n,x_{n+1})+d(x_n,x_n)}{d(x_n,x_{n+1})\cdot d(x_n,x_n)+k}\right)d(x_{n-1},x_n) \end{aligned}$$

Let $\beta_n = \left(\frac{d(x_n,x_{n+1})+d(x_n,x_n)}{d(x_n,x_{n+1})\cdot d(x_n,x_n)+k}\right)$

We get that, $d(x_n,x_{n+1}) \le \beta_n d(x_{n-1},x_n) \le \beta_n\beta_{n-1} d(x_{n-2},x_{n-1})$

$$d(x_n,x_{n+1}) \le \beta_n\beta_{n-1}\ldots\beta_1 d(x_0,x_1).$$

Observe that, $\{\beta_n\}$ is a non-increasing sequence, with positive terms.

So, $\beta_n\beta_{n-1}\ldots\beta_1 \le \beta_1^n$ and $\beta_1^n \to 0$ as, $n \to \infty$.

It follows that, $\lim_{n\to\infty} \beta_n\beta_{n-1}\ldots\beta_1 = 0$

Thus we get,

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$$

Now for all $m,n \epsilon \mathbb{N}$ and $m > n$

We have,

$$d(x_m,x_n) \le s^{m-n}d(x_m,x_{m-1}) + s^{m-n-1}d(x_{m-1},x_{m-2}) + \cdots s^2 d(x_{n+2},x_{n+1}) + sd(x_{n+1},x_n)$$

$$\begin{aligned} d(x_m,x_n) \le{}& s^{m-n}\beta_{m-1}\beta_{m-2}\ldots\beta_1 d(x_0,x_1) + s^{m-n-1}\beta_{m-2}\beta_{m-3}\ldots\beta_1 d(x_0,x_1) + \cdots + s^2\beta_{n+1}\beta_n\ldots\beta_1 d(x_0,x_1) \\ &+ s\beta_n\beta_{n-1}\ldots\beta_1 d(x_0,x_1) \end{aligned}$$
$$d(x_m,x_n) \le s^{m-n}\beta_1^{m-1}d(x_0,x_1) + s^{m-n-1}\beta_1^{m-2}d(x_0,x_1) + \cdots + s^2\beta_1^{n+1}d(x_0,x_1) + s\beta_1^n d(x_0,x_1)$$
$$d(x_m,x_n) \le (s^{m-n}\beta_1^{m-1} + s^{m-n-1}\beta_1^{m-2} + \cdots + s^2\beta_1^{n+1} + s\beta_1^n)d(x_0,x_1)$$

Take $n \to \infty$ we get, $d(x_m,x_n) \to 0$

Similarly we can prove, $d(x_n,x_m) \to 0$

Thus $\{x_n\}$ is a Cauchy sequence.

Now, since (X,d) is a complete dqb-metric space, the sequence $\{x_n\}$ must converge to some point x^* in dqb-metric space (X,d), Such that $x_n \to x^*$ as $n \to \infty$.

i.e. $\lim_{n\to\infty} d(x_n,x^*) = 0 = \lim_{n\to\infty} d(x^*,x_n)$

$$d(Tx^*,x^*) \le sd(Tx^*,Tx_n) + sd(Tx_n,x^*)$$
$$d(Tx^*,x^*) \le s\left(\frac{d(x_n,Tx_n)+d(x_n,Tx^*)}{d(x_n,Tx_n)\cdot d(x_n,Tx^*)+k}\right)d(x^*,x_n) + sd(Tx_n,x^*)$$
$$d(Tx^*,x^*) \le s\left(\frac{d(x_n,x_{n+1})+d(x_n,Tx^*)}{d(x_n,x_{n+1})\cdot d(x_n,Tx^*)+k}\right)d(x^*,x_n) + sd(x_{n+1},x^*)$$

Taking limit as $n \to \infty$ on both sides we get,

$$d(Tx^*,x^*) \le s\cdot 0 + s\cdot 0$$
$$d(Tx^*,x^*) \le 0 \Longrightarrow d(Tx^*,x^*) = 0$$

And similarly we can prove that, $d(x^*,Tx^*) = 0$

Giving that, $Tx^* = x^*$

This proves that x^* is fixed point of T
Uniqueness:

Consider two fix points x* and y* of T. Therefore,$Tx^* = x^*$ and $Ty^* = y^*$.
Consider,

$$d(x^*,y^*) = d(Tx^*,Ty^*) \leq \left(\frac{d(y^*,Ty^*) + d(y^*,Tx^*)}{d(y^*,Ty^*) \cdot d(y^*,Tx^*) + k}\right) d(x^*,y^*)$$

$$d(x^*,y^*) \leq 0 \Longrightarrow d(x^*,y^*) = 0$$

And $d(y^*,x^*) = d(Ty^*,Tx^*) \leq \left(\frac{d(x^*,Tx^*)+d(x^*,Ty^*)}{d(x^*,Tx^*)\cdot d(x^*,Ty^*)+k}\right) d(y^*,x^*)$

$$d(y^*,x^*) \leq 0 \Longrightarrow d(y^*,x^*) = 0$$

Thus $d(x^*,y^*) = 0$ and $d(y^*, x^*) = 0$ implies that, $x^* = y^*$
Hence x^* is a unique fixed point of T.
(ii)
Now,
Consider,$d(T^n x^*,x^*) = d(T^{n-1}(Tx^*),x^*) = d(T^{n-1}x^*,x^*) = \cdots = d(Tx^*,x^*) = 0$
$i.e\ d(T^n x^*,x^*) = 0$ Similarly we can prove that$d(x^*,T^n x^*) = 0$
This gives that, $T^n x^* = x^*$
Hence $\{T^n x^*\}$ converges to a fixed point, for all $x^* \epsilon X$.

2.6 Corollary: Let (X,d) be a complete dqb-metric space with coefficient s. Suppose the mapping $T: X \to X$ satisfies the following conditions:

$$d(Tx,Ty) \leq \left(\frac{d(y,Ty) + d(y,Tx)}{d(y,Ty) \cdot d(y,Tx) + 1}\right) d(x,y)$$

For all $x,y \,\epsilon\, X$, then
(i) T has a unique fixed point in X.
(ii) $\{T^n x^*\}$ Converges to a fixed point, for all $x^* \,\epsilon\, X$.

Proof: The proof follows from the proof of the above theorem by putting k=1.

2.7 Theorem: Let (X,d) be a complete dqb-metric space with coefficient s. Suppose that mapping $T: X \to X$ satisfies the following condition:

$$d(Tx,Ty) \leq \left(\frac{d(y,Ty) + d(y,Tx)}{d(y,Ty) \cdot d(y,Tx) + k}\right) d(x,y)$$

For all $x,y \,\epsilon\, X$, where $k \geq 1$then
(iii) T has a unique fixed point in X.
(iv) $\{T^n x^*\}$ Converges to a fixed point, for all $x^* \,\epsilon\, X$.

Proof: (i) Let $x_0 \,\epsilon\, X$ be arbitrary point in X and choose a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\leq \left(\frac{d(x_n,Tx_n) + d(x_n,Tx_{n-1})}{d(x_n,Tx_n) \cdot d(x_n,Tx_{n-1}) + k}\right) d(x_{n-1},x_n) \\ &\leq \left(\frac{d(x_n,x_{n+1}) + d(x_n,x_n)}{d(x_n,x_{n+1}) \cdot d(x_n,x_n) + k}\right) d(x_{n-1},x_n) \end{aligned}$$

Let $\beta_n = \left(\frac{d(x_n,x_{n+1})+d(x_n,x_n)}{d(x_n,x_{n+1})\cdot d(x_n,x_n)+k}\right)$

We get that,

$$d(x_n,x_{n+1}) \leq \beta_n d(x_{n-1},x_n) \leq \beta_n\beta_{n-1} d(x_{n-2},x_{n-1})$$

$$d(x_n,x_{n+1}) \leq \beta_n\beta_{n-1} \ldots \beta_1 d(x_0,x_1).$$

Observe that, $\{\beta_n\}$ is a non-increasing sequence with positive terms.
So, $\beta_n\beta_{n-1} \ldots \beta_1 \leq \beta_1^n$ and $\beta_1^n \to 0$ as, $n \to \infty$.
It follows that, $\lim_{n\to\infty} \beta_n\beta_{n-1} \ldots \beta_1 = 0$
Thus we get,

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$$

Now for all $m,n \,\epsilon\, \mathbb{N}$ and $m > n$
We have,

$$d(x_m,x_n) \leq s^{m-n} d(x_m,x_{m-1}) + s^{m-n-1} d(x_{m-1},x_{m-2}) + \cdots s^2 d(x_{n+2},x_{n+1}) + s d(x_{n+1},x_n)$$

$$d(x_m,x_n) \leq s^{m-n}\beta_{m-1}\beta_{m-2} \ldots \beta_1 d(x_0,x_1) + s^{m-n-1}\beta_{m-2}\beta_{m-3} \ldots \beta_1 d(x_0,x_1) + \cdots + s^2\beta_{n+1}\beta_n \ldots \beta_1 d(x_0,x_1) + s\beta_n\beta_{n-1} \ldots \beta_1 d(x_0,x_1)$$

$$d(x_m,x_n) \leq s^{m-n}\beta_1^{m-1} d(x_0,x_1) + s^{m-n-1}\beta_1^{m-2} d(x_0,x_1) + \cdots + s^2\beta_1^{n+1} d(x_0,x_1) + s\beta_1^n d(x_0,x_1)$$

$$d(x_m,x_n) \leq (s^{m-n}\beta_1^{m-1} + s^{m-n-1}\beta_1^{m-2} + \cdots + s^2\beta_1^{n+1} + s\beta_1^n) d(x_0,x_1)$$

Take $n \to \infty$ we get, $d(x_m,x_n) \to 0$
Similarly, we can prove, $d(x_n,x_m) \to 0$
Thus $\{x_n\}$ is a Cauchy sequence.

Now, since (X,d) is a complete dqb-metric space, the sequence $\{x_n\}$ must converge to some point x^* in dqb-metric space (X,d), Such that $x_n \to x^*$ as $n \to \infty$.

i.e. $\lim_{n\to\infty} d(x_n, x^*) = 0 = \lim_{n\to\infty} d(x^*,x_n)$

$$d(Tx^*,x^*) \leq sd(Tx^*,Tx_n) + sd(Tx_n,x^*)$$

$$d(Tx^*,x^*) \leq s\left(\frac{d(x_n,Tx_n)+d(x_n,Tx^*)}{d(x_n,Tx_n)\cdot d(x_n,Tx^*)+k}\right)d(x^*,x_n)+sd(Tx_n,x^*)$$
$$d(Tx^*,x^*) \leq s\left(\frac{d(x_n,x_{n+1})+d(x_n,Tx^*)}{d(x_n,x_{n+1})\cdot d(x_n,Tx^*)+k}\right)d(x^*,x_n)+sd(x_{n+1},x^*)$$

Taking limit as $n \to \infty$ on both sides we get,

$$d(Tx^*,x^*) \leq s\cdot 0 + s\cdot 0$$
$$d(Tx^*,x^*) \leq 0 \Longrightarrow d(Tx^*,x^*) = 0$$

And similarly we can prove that, $d(x^*,Tx^*) = 0$
Giving that, $Tx^* = x^*$
This proves that x^* is fixed point of T
Uniqueness:

Consider two fix points x* and y* of T. Therefore,$Tx^* = x^*$ and $Ty^* = y^*$.
Consider,

$$d(x^*,y^*) = d(Tx^*,Ty^*) \leq \left(\frac{d(y^*,Ty^*)+d(y^*,Tx^*)}{d(y^*,Ty^*)\cdot d(y^*,Tx^*)+k}\right)d(x^*,y^*)$$
$$d(x^*,y^*) \leq 0 \Longrightarrow d(x^*,y^*) = 0$$

And $d(y^*,x^*) = d(Ty^*,Tx^*) \leq \left(\frac{d(x^*,Tx^*)+d(x^*,Ty^*)}{d(x^*,Tx^*)\cdot d(x^*,Ty^*)+k}\right)d(y^*,x^*)$

$$d(y^*,x^*) \leq 0 \Longrightarrow d(y^*,x^*) = 0$$

Thus $d(x^*,y^*) = 0$ and $d(y^*, x^*) = 0$ implies that, $x^* = y^*$
Hence x^* is a unique fixed point of T.
(ii)
Now,
Consider,$d(T^nx^*,x^*) = d(T^{n-1}(Tx^*),x^*) = d(T^{n-1}x^*,x^*) = \cdots = d(Tx^*,x^*) = 0$
$i.e\ d(T^nx^*,x^*) = 0$ Similarly we can prove that$d(x^*,T^nx^*) = 0$
This gives that, $T^nx^* = x^*$
Hence $\{T^nx^*\}$ converges to a fixed point, for all $x^* \epsilon X$.

2.8 Corollary: Let (X,d) be a complete dqb-metric space with coefficient s. Suppose that mapping $T: X \to X$ satisfies the following condition:

$$d(Tx,Ty) \leq \left(\frac{d(y,Ty)+d(y,Tx)}{d(y,Ty)\cdot d(y,Tx)+1}\right)d(x,y)$$

For all $x,y\ \epsilon\ X$, then

(v) T has a unique fixed point in X.
(vi) $\{T^nx^*\}$ Converges to a fixed point, for all $x^*\ \epsilon\ X$.

Proof: The proof follows from the proof of the above theorem by putting k=1.

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