SEMIGENERALIZED α-COMPACT SPACES

S. VONITH KUMAR* AND P. GOMATHI SUNDARI#
* Department of Mathematics, Swami Dayananda college of Arts and Science, Manjakudi, Thanjavur, Tamilnadu, India.
#Department of Mathematics, Rajah Serfoji Government College, Thanjavur, Tamilnadu, India.

***Abstract* – In this paper, we introduce and study the notion of compact spaces in topological spaces via semigeneralized α-open sets. Several properties of these notions are discussed.**

I. INTRODUCTION

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms, compactness, connectedness etc. by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of sgα-open sets was introduced by Rajesh and Biljana in 2009. In this paper we introduce and study some topological properties of sgα-convergence and sgα-cluster points of net and filter by using the concept of sgα-open sets. Throughout this paper (X,τ) and (Y,σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. In this paper, we introduce and study the notion of compact spaces in topological spaces via semigeneralized α-open sets. Several properties of these notions are discussed.

II. PRELIMINARIES

Throughout this paper, (X, τ) denote topological spaces. The closure of S ⊆ X and the interior of S denoted with C1(S) and Int(S), respectively.

Definition 2.1. A subset A of a topological space (X, τ) is said to be semi-open [1] (resp. α-open [2]) if A ⊆ C1(Int (A)) (resp. A ⊆ Int (C1(Int (A)))).

Definition 2.2. The intersection of all α-closed sets of a topological space (X,T) containing A⊆ X is called the α-closure of A and is denoted by α C1 (A) [2].

Definition 2.3 A subset A of a topological space (X, τ) is called semigeneralized α-closed (briefly sgα-closed) [3] if α C1(A) ⊆ U whenever A ⊆ U and U is semiopen in (X, τ). The complement of sgα-closed set is called sgα-open and class of all sgα-open (resp. sgα-closed) subsets of (X, τ) is denoted by sgαO(X) (resp. sgαC(X)).

Definition 2.4. The intersection of all sgα-closed sets containing A is called the sgα-closure of A [3] and is denoted by sgα C1(A).

III. SGα-COMPACT TOPOLOGICAL SPACES

Definition 3.1. A collection {A_i : i∈ Δ} of sgα-open sets in a topological space (X,τ) is called a sgα-open cover of a subset A in X if A $\subset \bigcup_{i\in\Delta} A_i$.

Definition 3.2. A topological space (X, τ) is said to be sgα-compact if for each sgα-open cover of X has a finite subcover.

*Example 3.3***.** Let X = R, τ_n = $\{\Phi, \mathbb{R}\}\cup\{(a,\infty) : a \in \mathbb{R}\}$. It is clear that SGαO (X, τ_n) = $\{\Phi, \mathbb{R}\}\cup\{(a,\infty) : a \in \mathbb{R}\}$. Clearly, the topological space (X, τ) is not a sgα-compact space.

However, we have the following Theorem

Theorem 3.4. Every finite topological space (X, τ) is sgα-compact.

Proof: Let X = $\{x_1, x_2, x_3, \dots x_n\}$. Let C be a sgα-open covering of X. Then each element in X belongs to one of the members of C, say, $x_1 \in G_1$, $X_2 \in G_2, \dots x_n \in G_n$, where $G_i \in$ C, i=1, 2, …,n. Then the collection $\{G_1, G_2, G_3, \dots G_n\}$ is a finite subcover of X. Hence (X, τ) is sgα-compact.

Theorem 3.5. Finite union of sgα-compact sets is sgα-compact.

Proof: Let U and V any two sgα-compact subset of X. Let C be a sgα-open cover of U∪V. Then C will also be a sgα-open cover of both U and V. So by hypothesis, there exists a finite subcollection of C of sgα-open sets, say $\{U_1, U_2, U_3, \dots U_n\}$ and $\{V_1, V_2, V_3, \dots V_n\}$ covering U and V respectively. Clearly, the collection $\{U_1, U_2, U_3, \dots U_n, V_1, V_2, V_3, \dots V_n\}$ is a finite collection of sgα-open sets covering U∪V. By induction, every finite union of sgα-compact sets is sgα-compact.

Theorem 3.6. Every sgα-closed subset of a sgα-compact space (X, τ) is sgα-compact.

Proof: If G is sgα-closed set in the sgα-compact space (X, τ) and C is any sgα-open cover of G, then the collection $C^* = (X \backslash G) \cup C$ is a sgα-open cover of X. Since X is sgα-compact, the collection C^* has finite subcover. If this finite subcover contains the set X\G, discard it, otherwise leave the finite sub cover alone, the resulting collection is a finite sub cover of C.

Theorem 3.7. Every infinite subset of a sgα-compact space (X, τ_n, γ) has at least one sgα-cluster point in X.

Proof: Suppose X is sgα-compact space and let A be an infinite subset of X. Assume that A has no sgα-cluster points in X. Then for each x ∈ X, there exists a sgα-open set U_x such that $U_x \cap A = \{x\}$ or Φ. Now the collection $\{U_x : x \in X\}$ is a sgα-open covering of X. Since X is sgα-compact, there exists points $x_1, x_2, \ldots, x_n$ in X such that $\bigcup_{i=1}^{n} U_{x_i} = X$. But $(U_{x_1} \cap A) \cup (U_{x_2} \cap A) \cup \ldots (U_{x_n} \cap A)$ = $\{x_1\} \cup \{x_2\} \cup \ldots \{x_n\}$ or Φ. It follows that $(U_{x_1} \cup U_{x_2} \cup \ldots U_{x_n}) \cap A = \{x_1, x_2, \ldots, x_n\}$ or Φ. Hence $A = \{x_1, x_2, \ldots, x_n\}$ or Φ, contradicts that A is infinite.

Definition 3.8. A collection η of P(X) is said to satisfy the finite intersection condition if for every finite subcollection $\{F_1, F_2, \ldots F_n\}$ of η, the intersection $F_1 \cap F_2 \cap \ldots \cap F_n$ is nonempty.

We will give several characterizations of the sgα-compact spaces. The first characterization makes use of the finite intersection condition.

*Theorem 3.9***.** The following statements are equivalent for any topological space (X, τ_n, γ):

(1). (X, τ) is sgα-compact.
(2). Given any family Fof sgα-open sets, if no finite subfamily of F covers X, then F does not cover X.
(3). Given any family F of γ sgα-closed sets, if F satisfies the finite intersection condition, then $\bigcap\{A : A \in F\} \neq \Phi$.
(4). Given any family F of subsets of X, if F satisfies the finite condition, then $\bigcap\{sg\alpha cl(A) : A \in F\} \neq \Phi$.

Proof: (1) <=>(2) and (2) <=> (3) are obvious.

(3) <=> (4): If F⊂P(X) satisfies the finite intersection condition, then $\bigcap\{sg\alpha cl(A) : A \in F\}$ is a family of sgα-closed sets, which obviously satisfies the finite intersection condition.

(4)<=>(3):Follows from the fact that A= sgαcl(A) for every sgα-closed subset A of X.

Theorem 3.10. A topological space (X, τ) is sgα-compact if and only if for every collection of sgα-closed sets in X, satisfying the finite intersection condition, the intersection of all the sgα-closed sets in the condition is nonempty.

Proof: Suppose Γ is the collection of subsets of X. Let ξ = $\{X\Gamma : G \in \varsigma\}$ be the collection of their compliments. Then we have the following equivalent statements:

(i). Γ is collection of all sgα-open sets if and only if ξ is collection of all sgα-closed sets in X.
(ii). The collection Γ is a sgα-open cover of X if, and only if the intersection of all sgα-closed elements in ξ is empty.
(iii). The finite subcollection $\{G_1, G_2, \ldots G_n\}$ of Γ is a sgα-open cover of X if, and only if the intersection of the corresponding sgα-closed elements in ξ is nonempty.

Now X is sgα-compact if, and only if given any collection Γ of sgα-open sets, if Γ is sgα-open cover of X if, and only if any given collection Γ of sgα-open sets, if there is no finite sub collection of Γ covers X, then Γ is not a sgα-open cover of X if, and only if any collection ξ of sgα-closed sets, if for every finite intersection of sgα-closed in ξ is nonempty, then the intersection f all the sgα-closed sets in ξ is nonempty.

Definition 3.11. Let (X, τ) be topological space. A point x ∈ X is said to be a sgα-cluster point of a net $\{x_\alpha\}_{\alpha\epsilon\Delta}$ if $\{x_\alpha\}_{\alpha\epsilon\Delta}$ is frequently in every sgα-open set containing x. we denote by sgα-$cp_\gamma\{x_\alpha\}_{\alpha\epsilon\Delta}$ the set of all sgα-cluster points of a net $\{x_\alpha\}_{\alpha\epsilon\Delta}$.

*Theorem 3.12.*The set of all sgα-cluster point of an arbitrary net in X is sgα-closed.

Proof: Let $\{x_\alpha\}_{\alpha\epsilon\Delta}$ be a net in X. Set A = sgα-$cp_\gamma\{x_\alpha\}_{\alpha\epsilon\Delta}$. Let x ∈ X\A. Then there exists a sgα-open set U_x containing x and $\alpha_x \epsilon \Delta$ such that $\beta_x \notin U_x$whenever $\beta \epsilon \Delta$, $\beta \geq \alpha_x$. It turns out that $U_x \subset X \backslash A$, hence x ∈ sgαint(X\A) = X\ sgαcl(A). This shows that sgαcl(A) ⊂ A; hence A is sgα-closed in (X, τ).

Theorem 3.13. A topological space (X, τ) is sgα-compact if, and only if each net $\{x_\alpha\}_{\alpha\epsilon\Delta}$ in X has atleast one sgα-compact point.

Proof: Let (X, τ) be a sgα-compact space. Assume that there exist some net $\{x_\alpha\}_{\alpha\epsilon\Delta}$ in X such that sgα-$cp_\gamma\{x_\alpha\}_{\alpha\epsilon\Delta}$ is empty. Then for every x ∈ X, there exist U(x) ∈ SGαO(X, x) and $\alpha(x) \in \Delta$ such that $x_\beta \notin U(x)$ when ever $\beta \geq \alpha(x), \beta\epsilon\Delta$. Then the family $\{U(x) : x \in X\}$ is cover of X by a sgα-open sets and has a finite subcover, say, $\{U_k : k = 1, 2, \ldots n\}$ where $U_k = U(x_k)$ for k = 1,2,….n, $\{x_k : k = 1,2, \ldots n\}$. For every $\beta\epsilon\Delta$ such that $\beta \geq \alpha$ we have, $x_\beta \notin U_k$, k = 1,2,….n, hence $x x_\beta \notin X$, which is a contradiction. Conversely, if X is not sgα-compact, there exists $\{U_i : i \in \Delta\}$ a cover of X by sgα-open sets, which have no finite subcover. Let F(Δ) be the family of all finite subsets of Δ. Clearly (F(Δ, ⊂) is a directed set. For each J ∈ F(Δ), we may choose x_j∈

X\∪$\{U_i : i \in j\}$. Let us consider the net $\{x_j\}_{j\epsilon F(\Delta)}$. By, hypothesis the set $sg\alpha$-cp$_\gamma\{x_j\}_{j\epsilon F(\Delta)}$ is nonempty. Let x ∈ sgα-cp$_\gamma\{x_j\}_{j\epsilon F(\Delta)}$ and let i_0∈Δ such that x ∈U_{i0}. B y the definition of sgα-cluster point for each J ∈F(Δ) there exists J* ∈ F(Δ) such that J⊂J* and x_{j^*}∈ U_{i0}. For j = $\{i_0\}$, there exists j*∈ F(Δ) such that i_0∈ j* and x_{j^*}∈ U_{i0} . But x_{j^*}∈X\∪$\{U_i : i \in j^*\} \subset X\backslash U_{i0}$. The contradiction we obtained shows that (X, τ) is sgα-compact.

In the following, we will give a characterization of sgα-compact spaces by means of filterbase.

Let us recall that a nonempty family F of subsets of X is said to be a filterbase on X if Φ ∉F and each intersection of two members of F a third member of F. Notice that each chain in the family of all filterbase on X (ordered by inclusion) has an upper bound, for example the union of all members of the chain. Then by, zorn's lemma, the family of all filterbase on X has atleast one maximal element. Similarly, the family of all filterbase on X containing a given filterbase F has at least one maximal element.

Definition 3.14. A filterbase F on a topological space (X, τ) is said to be

(1). sgα-coverage to a point x ∈ X if for each sgα-open set U containing x, there exists B ∈F such that B ⊂ U.
(2). sgα-accumulate at x ∈ X if U ∩ B ≠ Φ for every sgα-open set U containing x and every B ∈F.

Remark 3.15. A filterbase F sgα-accumulates at x if, and only if x∈∩{sgα$cl(B)$: $B \in F\}$. Clearly, if a filterbase F sgα-converges to x ∈ X, then F sgα-accumulates at x.

Lemma 3.16. If a maximal filterbase F sgα-accumulates at x ∈ X, then F sgα-converges to x.

Proof: Let F be a maximal filterbase which sgα-accumulates at x ∈ X. if F does not γ sgα-converges to x, then there exists a sgα-open set U containing x such that U ∩ B ≠ Φ and (X\U)∩B ≠ Φ for every B ∈F. Then F∪$\{U \cap B : B \in F\}$ is a filterbase which strictly contains F, which is a contradiction.

Theorem 3.17. For a topological space (X, τ) the following statements are equivalent:

(1). (X, τ) is sgα-compact.
(2). Every maximal filterbase sgα-converges to some point of X.
(3). Every filterbase sgα-accumulates at some point of X.
(4). For every family $\{F_\alpha : \alpha\epsilon\Delta\}$ of sgα-closed subsets of (X, τ), such that $\cap\{F_\alpha : \alpha\epsilon\Delta\} = \emptyset$ there exists a finite subset Δ_0 of Δ such that$\{F_\alpha : \alpha\epsilon\Delta_0\} = \emptyset$.

Proof: (1)=> (2): Let F_0 be a maximal filterbase on X. suppose that F_0 does not sgα-converge to any point of X. Then by Lemma 3.16, F_0 does not sgα-accumulate at any point of X. For each x∈X, there exists a sgα-open set Ux containing x and Bx∈F0 such that U_x∩B_x = Φ. The family $\{U_x : x \in X\}$ is cover of X by sgα-open sets. By (1), there exists a finite subset $\{x_1, x_2, \dots x_n\}$ of X such that X = $\bigcup\{U_{xk} : k = 1,2, \dots n\}$. Since F_0 is a filterbase, there exists B_0∈F_0 such that B_0 = Φ. This is a contradiction.

(2)=>(3): Let F be a filterbase on X. There exists a maximal filterbase F_0 such that F⊂F_0. By (2), F_0 sgα-converges to some point x_0∈X. Let B∈F. For every U∈SGαO(X, x_0). There exists B_u∈F_0 such that B_u⊂U, hence U∩B ≠ Φ, since it contains the member B_u∩ of F_0. This shows that F sgα-ccumulates at x_0.

(3)=>(4): Let $\{F_\alpha : \alpha\epsilon\Delta\}$ be a family of sgα-closed subsets of (X, τ) such that $\cap\{F_\alpha : \alpha\epsilon\Delta\}$ = Φ. If possible suppose that every finite subfamily has a nonempty intersection. Then $\beta = \{\bigcap_{i=1}^{n} F_{\alpha_i} : n\,\epsilon N, F_{\alpha_i}\epsilon\{F_\alpha : \alpha\epsilon\Delta\}\}$ form a filterbase on X. Then by (3) β is sgα-accumulates to some points x∈X. this implies that for every sgα-open set U containing x, Fα∩U ≠ Φ for F_α∈β.and every $\alpha \in \Delta$. Since x∉∩$\{F_\alpha : \alpha\epsilon\Delta\}$, there exists $\alpha_0\epsilon\Delta$ such that x∉F_{α_0}. Therefore x∈X\$F_{\alpha_0}$, which is sgα-open set in X. But $F_{\alpha_0} \cap X\backslash F_{\alpha_0} = \Phi$, then we get the contradiction of the fact that β is sgα-accumulates to x.

(4) => (1): Let $\{U_\alpha : \alpha\epsilon\Delta\}$ be a sgα-open cover of X. Then $\{X\backslash U_\alpha : \alpha\epsilon\Delta\}$ is a family of sgα-closed subset of X such that $\cap\{X\backslash U_\alpha : \alpha\epsilon\Delta\}$ = Φ. Then by (4) there exists a finite subset Δ_0 of Δ such that $\cap\{X\backslash U_\alpha : \alpha\epsilon\Delta\}$ = Φ. This implies that X = ∪$\{U_\alpha : \alpha\epsilon\Delta_0\}$; hence (X, τ) is a sgα-compact.

Definition 3.18. A point x in a topological space (X, τ) is said to be a sgα-accumulation point of a subset A of X if $|S \cap A| = |A|$ for each S∈ SGαO(X, x).

Definition 3.19. In a topological space (X, τ), a point x is said to be a sgα-adherent point of a filter base F on X if it lies in the sgα-closure of all sets of F.

Theorem 3.20. A topological space (X, τ) is sgα-compact if, and only if each infinite subset

Proof: Let (X, τ) be a sgα-compact space and A an infinite subset of X. Let K be the set of all points of x in X which are not sgα-complete accumulation points of A. Now it is obvious that for each point x in K, we are able to find U(x) ∈ SGαO(X, x) such that $|A \cap U(x)| \neq |A|$. If K is the whole space X, then C = $\{U(x): x \in X\}$ is a sgα-open cover of X. By the hypothesis X is sgα-compact, so there exists a finite subcover G = $\{U(x_i): i = 1,2, \dots n\}$ such that A ⊂ $\cup\{U(x_i) \cap A : i = 1,2, \dots n\}$. Then $|A| = \max\{|A \cap U(x_i)| : i = 1,2, \dots . n\}$ which does not agree with what we assumed. This implies that A has a sgα-complete accumulation point. Now assume that X is not sgα-compact and that every infinite subset A of X has a sgα-complete accumulation point in X. it follows that, there exists a sgα-open cover S with no finite subcover. Set $\alpha = min\{|\varphi| : \varphi \subset S, where\ \varphi\ is\ the$ sgα − open cover of X}. Fix $\varphi \subset$ S for which $|\varphi| = \alpha$ and $\cup\{U : U \in \varphi\} = X$. Then, by hypothesis $\alpha \geq |\aleph|$, where ℵ denotes the set of all natural numbers. By well ordering of φ by some minimal well ordering "~", suppose that U is any member of φ. By minimal well-ordering "~" we have $|\{V : V \in \varphi, V\sim U\}| < |\{V : V \in \varphi\}|$. Since φ can not have sub cover with cardinality less than α, then for each U∈ φ we have X≠∪$\{V : V \in \varphi, V\sim U\}$. For each U ∈ φ we have X ≠ ∪$\{V : V \in \varphi, V\sim U\}$. For each U∈$\varphi$, choose a point x (U)∈X\∪$\{V \cup \{x(V)\} : V \in \varphi, V\sim U\}$. We always able to do this if not one can choose a cover of smaller cardinality from φ. If H = $\{x(U): U \in \varphi\}$, then to finish the proof we will show that H has no sgα-complete accumulation point in X. suppose that z∈X. Since φ is a sgα-open cover of , z is a point of some set, say, W in φ. By the fact that U~W, we have x(U)∈W. it follows that T = $\{U : U \in \varphi\ and\ x(U) \in W\} \subset \{V : v \in \varphi, V \sim W\}$. This means two distinct points U and W in φ, we have x(U) ≠ x(W). This means that H has no sgα-complete accumulation point in X, which contradicts to our assumptions. Therefore, X is sgα-compact.

Theorem 3.21. A topological space (X, τ) is sgα-compact if, and only if every net in X with a well ordered directed set as its domain sgα-accumulation to some point of X.

Proof: suppose that X is sgα-compact and A = $\{x_\alpha : \alpha \in A\}$ a net with a well ordered directed set A as domain. Assume that A has no sgα-adherent point in X. Then for each x∈X, there exists V(x)∈SGαO(X, x) and an $\alpha(x) \in \wedge\ such\ that$ V(x)∩$\{x_\alpha : \alpha \geq \alpha(x)\}$ is a subset of a X\V(x). Then collection C = $\{V(x) : x \in X\}$ is a sgα-open cover of X. since X is sgα-compact, C has a finite subfamily $\{V(x_i): i = 1,2, \dots n\}$ such that X $=\cup_{i=1}^{n} V(x_i)$. Suppose that the corresponding elements of ∧ be a$\{\alpha(x_i)\}$, where i= 1,2,…n. since ∧ is well ordered and $\{\{\alpha(x_i)\} :$ i = 1,2, … n.$\}$ is finite, the largest element of $\{\alpha(x_i)\}$ exists. Suppose it is $\{\alpha(x_i)\}$. Then for $\beta \geq \{\alpha(x_i)\}$, we have $\{x_\delta : \delta \geq \beta\} \subset \cap_{i=1}^{n}\{X\backslash V(x_i)\} = X\backslash \cup_{i=1}^{n} V(x_i) = \emptyset$, which is impossible. This shows that A has at least one sgα-adherent point in X. Conversely, suppose that S is an infinite subset of X. According to Zorn's lemma, the finite set S is well ordered. This means that we can assume S to be a with domain which is well ordered index set. It follows that S has a sgα-adherent point z. Therefore, z is a sgα-complete accumulation point of S. This shows that X is sgα-compact.

Theorem 3.22. A topological space (X, τ) is sgα-compact if, and only if each filter base in X has atleast one sgα-adherent point.

Proof: Suppose that X is sgα-compact and F = $\{F_\alpha : \alpha \in \Delta\}$ a filter in it. Since all finite intersections of F_α's are nonempty, it follows that all finite intersections of sgα$cl(F_\alpha)$'s are also nonempty. By finite intersection property $\cap_{\alpha\in\Delta}$ sgα$cl(F_\alpha) \neq \Phi$. This implies that F has at least one sgα-adherent point. Now suppose that F is any family of sgα-closed sets. Let each finite intersection be nonempty, the set F_α with finite intersection establish the filter base F. Therefore, F sgα-accumulates to some point z in X. It follows that z∈$\cap_{\alpha\in\Delta} F_\alpha$. Now we have by theorem 3.9(3), X is sgα-compact.

Theorem 3.23. A topological space (X, τ) is sgα-compact if, and only if each filter base in X, with at most sgα-adherent point, is sgα-convergent.

Proof: Suppose that X is sgα-compact, $x \in X$ and F is a filter base on X. The sgα-adherent of F is a subset of $\{x\}$. Then the sgα-adherent of F is equal to {x} by Theorem 3.22. Assume that there exists V ∈ SGαO(X, x) such that for all F∈F, F∩(X\V)≠Φ. Then $\varphi = \{F\backslash V : F \in F\}$ is a filter base on X. Then sgα-adherent of φ is nonempty. However, $\cap_{F\in F}$ sgα$cl(F\backslash V) \subset (\cap_{F\in F}$ sgαcl$(F)) \cap (X\backslash V) = \{x\} \cap (X\backslash V) = \emptyset$, a contradiction. Hence for each V∈SGαO(X, x), there exists F∈F such that F⊂sgα$cl\ (V)$. This shows that F sgα-converges to x. Conversely, assume that F is a filter base on X with no sgα-adherent point. By hypothesis, F sgα-converges to some point z in X. suppose F_α is an arbitrary element of F. then for each V∈ SGαO(X, z), there exists a F_β∈F such that $F_\beta \subset V$. Since F is a filter base, there exists a δ such that $F_\delta \subset F_\alpha \cap F_\beta \subset F_\alpha \cap V$, where $F_\delta \neq \Phi$. This means that $F_\alpha \cap V \neq \Phi$ for every V∈ SGαO(X, z) and correspondingly for each $\alpha, z \in$ sgαcl(Fα). It follows that $z \in \cap_\alpha$ sgα$cl(F_\alpha)$. Therefore z is a sgα-adherent point of F, a contradiction. This shows that X is sgα-compact.

REFERENCES

[1] Levine, N, Semi-open sets and semi-continuity in topological spaces, Amer. math. maonthly, 70 (1963), 36-41.

[2] Njastad, O, On some classes of nearly open sets, Pacific j. Math., 15(1965), 961-970

[3] Rajesh, N and Biljana, K, Semigeneralized α-closed sets, Antartica J. Math., 6 (1) 2009, 1-12.