SOLUTION OF DUAL INTEGRAL EQUATIONS BY MELLIN TRANSFORM

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Abstract: The objective of this paper is to reduce a dual integral equation into an integral equation by using mellin transform whose kernel involves Generalized Polynomial function. We believe that definitely there are various possible ways to reduce such dual integral equations using different transform like those of Fourier, Henkel etc. For the reason of illustration we choose a dual integral equation of particular type and by use of fractional operator and mellin transform reduced it to an integral equation.

IndexTerms **– Generalized Polynomial Function; Mellin Transform; Fractional operators; Fox-H function.**

I- INTRODUCTION

Dual integral equations are often encountered in many branches of mathematical physics and they usually tale place while solving a boundary value problem with mixed boundary conditions In the present paper, attempt has been made to solve the certain dual integral equations involving Generalized Polynomial function as kernel by reducing them into an integral equations. Many attempts have been made in the past to solve such problems. The following integral equations are basic tool for our illustration.

$$\int_0^\infty k_1(x,u)\ A(u)\,du = \lambda(x);\ 0 \le x \le 1 \tag{1.1}$$

$$\int_0^\infty k_2(x,u)\ A(u)\,du = \omega(x);\ x > 1 \tag{1.2}$$

k_1 & k_2 are kernels defind over $x-u$ plane.

$$F_n^r\left(x,a,k,p\right) = x^{-a} e^{px^r} D^n \left\{x^{a+kn} e^{-px^r}\right\}\ a,k,p,r\ \ parameter.$$

II- THEOREM

If f is unknown function satisfying the dual integral equation.

$$\int_0^\infty (xy)^{a_1}\ e^{-(xy)^r}\ F_n^r(xy;a_1,k,1)\ f(y)\,dy\ = h(x);\ 0 \le x < 1 \tag{2.1}$$

$$\int_0^\infty (xy)^{a_2}\ e^{-(xy)^r}\ F_n^r(xy;a_2,k,1)\ f(y)\,dy = g(x);1 \le x < \infty \tag{2.2}$$

When h and g are prescribed function and a_1, a_2 *and* r are parameters, then f is giving by

$$f(x) = \frac{1}{r}\int_0^\infty L(xy)\ t(y)\,dy$$

Where

$$L(x) = H_{2,1}^{1,0}\left[x\left|\begin{matrix}(1,1) & \left(\frac{1}{r}(a_1+(k-1)n+1)\right),\frac{1}{r}\\ (1+(k-1)n,1) & \end{matrix}\right.\right]$$

and

$$t(x) = h(x), 0 \le x < 1$$

$$t(x) = \frac{r\,x^{(k-1)n+a_1}}{\left|\left(\frac{1}{r}(a_2-a_1)\right)\right.}\int_0^\infty (v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{-(k-1)n-a_2+r-1}\,g(v)\ dv;1 \le x < \infty$$

III- MATHEMATICAL PRELIMINARY

To prove the theorems we shall use Mellin transformer and fractional integral operator.

$$f^*(s) = M[f(x);s] = \int_0^\infty f(x)x^{s-1}\ dx \tag{3.1}$$

When $s = \sigma + i\tau$ is a complex variable.

The inverse melling transform f(x) of $f^*(s)$ is given by $M^{-1}[f^*(s)] = f(x) = \frac{1}{2\pi i}\int_{\sigma+i\infty}^{\sigma+i\infty} f^*(s)\,x^{-s}\,ds$ (3.2)

By convolution theorem for mellin transform

$$M\left[\int_0^\infty k(xy)\ f(y)\,dy;s\right] = k^*(s)\ f^*(1-s)$$

$$\int_0^\infty k(xy)\, f(y)\, dy = M^{-1}\left[k^*(s)\, f^*(1-s); s\right] = \frac{1}{2\pi i}\int_L k^*(s)\, f^*(1-s)\, x^{-s}\, ds \tag{3.3}$$

When L is suitable contour.
Fractional integral operator

$$\tau(\alpha;\, \beta;\, r; w(x)) = \frac{r\, x^{-r\alpha+r-\beta-1}}{\Gamma(\alpha)}\int_0^\infty (x^r - v^r)^{\alpha-1} v^{\beta} w(v)\, dv \tag{3.4}$$

$$R(\alpha;\, \beta;\, r; w(x)) = \frac{r\, x^{\beta}}{\Gamma(\alpha)}\int_x^\infty (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1}\, w(v)\, dv \tag{3.5}$$

IV- SOLUTION

Now taking

$$k_i(x) = x^{a_i}\, e^{-x^r}\, F_n^r\,(x, a_i, k, 1),\ i = 1, 2$$

Then from Erdeeyi [11] We get $k_i^*(s) = \dfrac{\Gamma s\, \Gamma\left(\frac{1}{r}(s+(k-1)n+a_i)\right)}{r\,\Gamma(s-n)}$, $i = 1, 2$ (4.1)

Hence by use of (3.3),(2.1) & (2.2) can be written as

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\, \Gamma\left(\frac{1}{r}(s+(k-1)n+a_1)\right)}{\Gamma(s-n)} f^*(1-s)\, x^{-s}\, ds = h(x);\ 0 \le x < 1 \tag{4.2}$$

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\, \Gamma\left(\frac{1}{r}(s+(k-1)n+a_2)\right)}{\Gamma(s-n)} f^*(1-s)\, x^{-s}\, ds = g(x);\ 1 \le x < \infty \tag{4.3}$$

Now operating a (4.2) by the operator (3.5) we get

$$\frac{rx^{\beta}}{\Gamma\alpha}\cdot\frac{1}{2r\pi i}\int_L \frac{\Gamma s\, \Gamma\left(\frac{1}{r}(s+(k-1)n+a_2)\right)}{\Gamma(s-n)} f^*(1-s)\, x^{-s}\, ds \int_x^\infty (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv = \frac{rx^{\beta}}{\Gamma\alpha}\int_x^\infty (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

Now putting $v^r = \dfrac{x^r}{t}$ and simplifying we get

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\, \Gamma\left(\left(\frac{1}{r}(s+(k-1)n+a_2)\right)\right)}{\Gamma(s-n)}\cdot\frac{x^{-s}}{\Gamma\alpha}\int_0^1 (1-t)^{\alpha-1}.t^{\left(\frac{\beta+s}{r}-1\right)}\, dt\ \ f^*(1-s)ds = \frac{rx^{\beta}}{\Gamma\alpha}\int_x^\infty (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv \tag{4.4}$$

$$\Rightarrow \frac{1}{2r\pi i}\int_L \frac{\Gamma s\, \Gamma\left(\left(\frac{1}{r}(s+(k-1)n+a_2\right)\right)}{\Gamma\left((s-n)\right)}\cdot\frac{x^{-s}}{\Gamma\alpha}\,\frac{\Gamma\left(\frac{1}{r}(\beta+S)\right)\Gamma\alpha}{\Gamma\left(\alpha+\frac{1}{r}\beta+\frac{1}{r}s\right)}\, f^*\,(1-s)\, ds = \frac{rx^{\beta}}{\Gamma\alpha}\int_x^\infty (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

$$\Rightarrow \frac{1}{2r\pi i}\int_L \frac{\Gamma s\, \Gamma\left(\left(\frac{1}{r}(s+(k-1)n+a_2)\right)\right)}{\Gamma(s-n)}\, x^{-s}\, \frac{\Gamma\left(\frac{1}{r}(\beta+s)\right)}{\Gamma\left(\alpha+\frac{1}{r}\beta+\frac{1}{r}s\right)} \times f*(1-s)\, ds = \frac{rx^{\beta}}{\Gamma\alpha}\int_x^\infty (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

In equation (4.4), we put $\beta = (k-1)n + a_1$ and $\alpha = \dfrac{1}{r}(a_2 - a_1)$, so that (4.4) Changes to

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\, \Gamma\left(\left(\frac{1}{r}(s+(k-1)n+a_2\right)\right)}{\Gamma(s-n)}\, x^{-s}\, \frac{\Gamma\left(\frac{1}{r}(s+(k-1)n+a_1)\right)}{\Gamma\left(\frac{a_2}{r}-\frac{a_1}{r}+\frac{1}{r}((k-1)n+a_1)+\frac{1}{r}s\right)}\ \ f*(1-s)\, ds = \frac{r\, x^{(k-1)n+a_1}}{\Gamma\left(\left(\frac{1}{r}(a_2-a_1)\right)\right)}$$

$$\times\int_x^\infty (v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{(k-1)n-a_2+r-1} g(v)dv\ \ 1 \le x \le \infty$$

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\, \Gamma\left(\left(\frac{1}{r}(s+(k-1)n+a_1\right)\right)}{\Gamma\left((s-n)\right)}\, x^{-s}\ \ f^*(1-s)ds = \frac{r\, x^{(k-1)n+a_1}}{\Gamma\left(\left(\frac{1}{r}(a_2-a_1)\right)\right)}\int_x^\infty (v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{(k-1)n-a_2+r-1} g(v)dv\ ;1 \le x \le \infty \tag{4.5}$$

Now we write

$t(x) = h(x), \quad 0 \le x < 1$

and

$$t(x) = \frac{r\, x^{(k-1)n+a_1}}{\Gamma\left(\frac{1}{r}(a_2 - a_1)\right)} \int_x^{\infty} (v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{-(k-1)n-a_2+r-1} g(v)\, dv; 1 \le x < \infty \qquad (4.6)$$

Now from (4.2), (4.5), (4.6) we get

$$\frac{1}{2r\pi i} \int_L \frac{\Gamma s\, \Gamma\left(\frac{1}{r}(s+(k-1)n+a_1\right)}{\Gamma(s-n)} x^{-s} \; f^*(1-s)ds = t(x) \qquad (4.7)$$

Again using (3.3), (4.1) & (4.6) becomes

$$\int_0^{\infty} k_1(xy)\, f(y)\, dy = t(x);\; 0 \le x < \infty \qquad (4.8)$$

When $k_1(x) = x^{a_1} e^{-x^r} F_n^r(x, a_1, k, 1)$

Thus pair at dual integral equation (1.1) & (1.2) we have been reduced to single integral equation (4.8). Hence by mellin transform (4.8) can be written as –

$$k_1^*(s)\, f^*(1-s) = T^*(s) \qquad (4.9)$$

Where $k_1^*(s) = \dfrac{\Gamma s\, \Gamma\left(\frac{1}{r}(s+(k-1)n+a_1)\right)}{\Gamma(s-n)}$

and $T^*(s)$ is the mellin transform of t(x).

Now replacing s by (1–s) in (4.9)

$$f^*(s) = L^*(s)\, T^*(1-s) \qquad (4.10)$$

$$L^*(s) = \frac{1}{k^*(1-s)} = \frac{\Gamma\left(1-s+(k-1)n\right)}{\Gamma(1-s)\, \Gamma\left(\frac{1}{r}(1-s+(k-1)n+a_1)\right)}$$

By use of definition of H – function, we get the inverse transform $L(x)$ at $L^*(s)$ as

$$L(x) = H_{2,1}^{1,0}\left(x \left| \begin{matrix} (1,1)\left(\frac{1}{r}(a_1+(k-1)n+1)\right), \frac{1}{r} \\ (1+(k-1)n, 1) \end{matrix} \right. \right) \qquad (4.11)$$

Taking inverse mellin transform of (4.10)

$$f(x) = \int_0^{\infty} L(xy)\, t(y)\, dy$$

Hence using (4.11) we get

$$f(x) = \frac{1}{r} \int_0^{\infty} H_{2,1}^{1,0}\left(xy \left| \begin{matrix} (1,1)\left(\frac{1}{r}(a_1+(k-1)n+1)\right), \frac{1}{r} \\ (1+(k-1)n, 1) \end{matrix} \right. \right) t(y) dy$$

When t(y) is given by (4.6).

Hence proved the theorem.

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