A SIMPLEX ALGORITHM FOR PIECEWISE-LINEAR FRACTIONAL PROGRAMMING

[1]J. DOSS, [2]Dr. SHALINI

1.Research Scholar, Department of Mathematics, St. Peter's Institute of Higher Education and Research, Avadi, Ch-54.

2.Associate Professor, Department of Mathematics, St. Peter's Institute of Higher Education and Research, Avadi, Ch-54

ABSTRACT:

The simplex method for linear programming can be extended to permit the minimization of any convex separable piecewise-linear objective, subject to linear constraints. Part I of this paper has developed a general and direct simplex algorithm for piecewise-linear programming, under convenient assumptions that guarantee a finite number of basic solutions, existence of basic feasible solutions, and nondegeneracy of all such solutions. Part II now shows how these assumptions can be weakened so that they pose no obstacle to effective use of the piecewise-linear simplex algorithm. The theory of piecewise-linear programming is thereby extended, and numerous features of linear programming are generalized or are seen in a new light. An analysis of the algorithm's computational requirements and a survey of applications will be presented in Part III.

Key words: Linear programming, simplex methods, piecewise-linear programming,

Introduction:

The minimization of convex separable piecewise-linear functions has traditionally been regarded as an application of purely linear programming. A given piecewise- linear program has first been converted to an equivalent larger linear program, by any of several well-known transformations. Then the linear program has been solved either by a standard simplex algorithm, or by a somewhat specialized simplex algorithm designed to exploit certain features of the transformation. Piecewise-linear simplex algorithms represent a more direct approach to the problem of piecewise-linear programming. They operate directly on an untrans-formed, unenlarged representation of the piecewise-linear objective function and linear constraints. Specialized piecewise-linear simplex algorithms have become standard in certain applications, notably in phase one of the linear simplex method and in 11 estimation. More recently, general simplex algorithms for piecewise-linear programming have been outlined by Premoli , Rockafellar , Snyder and the author . This chapter builds upon previous work to develop and analyze a general, computationally practical piecewise-linear simplex algorithm.

1. Piecewise-linear functions

The exposition of the piecewise-linear case begins here, with the introduction of a terminology for the functions that serve as objectives of primal and dual piecewise- linear programs. Several fundamental properties of convex separable piecewise-linear functions are also stated below. Proofs can be devised by arguing directly from the definitions, or by specializing standard properties of convex functions.

Separable piecewise-linear functions :

A function $f_k$of a single real variable x_k is piecewise-linear (or $P-L$) if it is linear on each of a series of intervals that partition the real line. To describe such a function it is convenient to define an increasing sequence of breakpoints, (y_k^h):

$. \quad . \quad . \gamma_k^{(-2)} < \gamma_k^{(-1)} < \gamma_k^{(0)} < \gamma_k^{(2)} < \gamma_k^{(2)} < . \quad . \quad .$

Then f_k is specified by a linear function on each interval $[\gamma_k^{(h)}, \gamma_k^{(h+1)}]$. The sequence $\gamma_k^{(h)}$ is potentially infinite in both directions, but it may effectively start with some $\gamma_k^{(s)} = -\infty$ = -co or may end with some $\gamma_k^{(t)} = -\infty$.

A separable piecewise-linear function f of n variables $x_1, x_2, \dots, x_n$. is the sum of n piecewise-linear functions f_k of these variables. Thus a separable $P-L$ function is defined by n increasing sequences of breakpoints $\gamma_k^{(h)}$, along with linear functions on all of the intervals $[\gamma_k^{(h)}, \gamma_k^{(h+1)}]$.

2. Continuous separable piecewise-linear functions

If the piecewise-linear function f_k is continuous, then it can be specified almost entirely in terms of its slopes on the intervals between breakpointsLet the slope of f_k be $c_k^{(h)}$ on the interval $[\gamma_k^{(h)}, \gamma_k^{(h+1)}]$.

Then for scalar values $u < v$ such that

$$\gamma_k^{(s)} \le u$$

$$\le \gamma_k^{(s+1)}$$

Thus any continuous $P-L$ function can be specified up to an additive constant by just the sequences $\gamma_k^{(h)}$ and $c_k^{(h)}$. Nothing is lost, moreover, in assuming that $c_k^{(h-1)} \neq c_k^{(h)}$ for all h, since $c_k^{(h-1)} = c_k^{(h)}$ implies that the breakpoint $\gamma_k^{(h)}$ is superfluous.

The sum of continuous $P-L$ functions of $x_1, x_2, \ldots, x_n$. is necessarily a continuous separable piecewise-linear function. It is specified up to an additive constant by the sequences $\gamma_k^{(h)}$ and $c_k^{(h)}$ for $k = 1, \ldots, n$. This is a sufficient specification for the purposes of piecewise-linear programming, which is concerned only with minimizing such functions. The above observations suggest a concise notation for continuous piecewise-linear functions in terms of their slopes and breakpoints will be written as

$[c/\gamma]_k x_k$

A continuous $P-L$ function of X_k, defined by slopes $c_k^{(h)}$ and breakpoints $\gamma_k^{(h)}$, will be written

$\left[\frac{c}{\gamma}\right] x = \sum_{k=1}^{n} [c/\gamma]_k x_k$

The piecewise-linear 'cost vector' $\left[\frac{c}{\gamma}\right]$ plays the role of the familiar cost vector c in linear programming, but the value of each 'coefficient' $\left[\frac{c}{\gamma}\right] x$ may depend on x_k.

3. Convex separable piecewise-linear functions

Consider now the continuous piecewise-linear functions $[c/\gamma]_k$ whose slopes, as well as their breakpoints, are increasing:

$. \quad . \quad . c_k^{(-2)} < c_k^{(-1)} < c_k^{(0)} < c_k^{(2)} < c_k^{(2)} < . \quad . \quad .$

Like the sequence of breakpoints, this sequence of slopes is potentially infinite.

However, it is useful to let the slopes begin with some $c_k^s = -\infty$ or end with some $c_k^s = \infty$; if $[c/\gamma]_k$ is required to be continuous only on some closed interval where it is finite, then there is a natural interpretation of infinite slopes in terms of infinite values of $[c/\gamma]_k$:

$c_k^s = -\infty$ implies $[c/\gamma]_k x_k = \infty$ for all $x_k < \gamma_k^{(S+1)}$

$c_k^t = \infty$ implies $[c/\gamma]_k x_k = \infty$ for all $x_k > \gamma_k^{(t)}$

The interval on which $[c/\gamma]$ is finite must be nonempty provided that at least one slope or one breakpoint is finite. Under this extended interpretation of $[c/\gamma]$, increasing sequences of slopes and breakpoints characterize the convex piecewise-linear functions.

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Property 1.

$[c/\gamma]_k$ is convex if and only if $c_k^{(h-1)} < c_k^{(h)}$ < for every finite breakpoint $\gamma_k^{(h)}$.

There is indeed a one-to-one correspondence between complementary increasing sequences of slopes and breakpoints, among which at least one slope or breakpoint is finite, and classes of convex $P-L$ functions that are proper (not everywhere infinite), closed (continuous where they are finite), and identical up to an additive constant. Henceforth $[c/\gamma]_k$ will denote only proper closed convex $P-L$ functions, which are the natural objective functions of piecewise-linear programs. Because $[c/\gamma]_k$ is convex, any tangent to its graph must lie below the graph. In more precise terms:

Property 2.

If $[c/\gamma]_k$is convex, then$[c/\gamma]_k$

for any real values of u and v such that $\gamma_k^{(h)} \leq u \leq \gamma_k^{(h+1)}$ and$-\infty < c_k^t < \infty$,in proving that the piecewise-linear simplex algorithm reduces the objective value at each iteration.

If $[c/\gamma]_k x_k$ is convex for each k, then $[\frac{c}{\gamma]_k} \sum_{k=1}^{n} [c/\gamma]_k x_k$ must also be convex. Hence n different increasing sequences of slopes and breakpoints γ_k^h define a (proper, closed) convex separable piecewise-linear function $[c/\gamma]$ of n variables.

4. Basic solutions

The basis matrix B for a piecewise-linear program is defined by m independent columns of the constraint matrix A, just as for a bounded-variable linear program.

However, whereas a basic solution is determined for an LP by fixing each nonbasic variable at one of its bounds, a basic solution for a $P-LP$ is determined by fixing each nonbasic variable at one of its breakpoints in the $P-L$ objective function.A basic solution for a $P-LP$ is defined by a basis matrix B together with an assignment of finite breakpoints γ_{Nj}^h to the nonbasic variables x_{Nj}. Writing simply γ_{Nj} for the breakpoint assigned to x_{Nj}., the formulas for a basic solution are

$\bar{x}_{Nj} = \gamma_{Nj}$ for all j, $B\bar{x}_B b - \sum_j a_{Nj}\gamma_{Nj}$

These expressions reduce when γ_{Nj} is taken to be $\gamma_{\overline{Nj}}$

At any particular basic solution, each basic variable has a certain slope in the objective function. The slope of x_{Bi} is given by

$$c_{Bi} = c_{Bi}^{(h)} \text{ where } \gamma_{Bi}^{(h)} \le x_{Bi} \le \gamma_{Bi}^{(h+1)}$$

The m-vector of these basic slopes, denoted c_B , plays the same role as the vector of basic costs.

In describing an iteration of the piecewise-linear simplex algorithm, it will be convenient to have a notation for the slopes and breakpoints of the objective function relative to the current basic solution• Thus, for each nonbasic variable x_{Nj}, Let $c_{Nj}^{+\{1\}}$, $c_{Nj}^{+\{2\}}$,… be the sequence of slopes as x_{Nj}. increases, and let $\gamma_{Nj}^{+\{1\}}$ $\gamma_{Nj}^{\{2\}}$,... be the intervening sequence of breakpoints; similarly, let $c_{Nj}^{-\{1\}}$, $c_{Nj}^{-\{2\}}$,… and $\gamma_{Nj}^{-\{1\}}$ $\gamma_{Nj}^{-\{2\}}$,... be the sequences of slopes and breakpoints as x_{Nj}. decreases.

The slopes of x_{Nj}. in successive intervals are then given as follows:

Interval	Slope of x_{Nj}
$[\gamma_{Nj}^{-\{h\}}, \gamma_{Nj}^{-\{h-1\}}]$	$c_{Nj}^{-\{h\}}$
. . .	
$[\gamma_{Nj}^{-\{2\}}, \gamma_{Nj}^{-\{1\}}]$	$c_{Nj}^{-\{2\}}$
$[\gamma_{Nj}^{-\{1\}}, \gamma_{Nj}]$	$c_{Nj}^{-\{1\}}$
$[\gamma_{Nj}, \gamma_{Nj}^{+\{1\}}]$	$c_{Nj}^{+\{1\}}$
$[\gamma_{Nj}^{+\{1\}}, \gamma_{Nj}^{+\{2\}}]$	$c_{Nj}^{+\{2\}}$
. . .	
$[\gamma_{Nj}^{+\{h-1\}}, \gamma_{Nj}^{+\{h\}}]$	$c_{Nj}^{+\{h\}}$

A complementary notation describes the slopes and breakpoints for each basic varia

$[\gamma_{Bi}^{-\{h+1\}}, \gamma_{Bi}^{-\{h\}}]$	$c_{Bi}^{-\{h\}}$
. . .	
$[\gamma_{Bi}^{-\{3\}}, \gamma_{Bi}^{-\{2\}}]$	$c_{Bi}^{-\{2\}}$
$[\gamma_{Bi}^{-\{2\}}, \gamma_{Bi}^{-\{1\}}]$	$c_{Bi}^{-\{1\}}$
$[\gamma_{Bi}^{-\{1\}}, \gamma_{Bi}^{+\{1\}}]$	c_{Bi}
$[\gamma_{Bi}^{+\{1\}}, \gamma_{Bi}^{+\{2\}}]$	$c_{Bi}^{+\{1\}}$
$[\gamma_{Bi}^{+\{2\}}, \gamma_{Bi}^{+\{3\}}]$	$c_{Bi}^{+\{2\}}$
. . .	
$[\gamma_{Bi}^{+\{h\}}, \gamma_{Bi}^{+\{h+1\}}]$	$c_{Bi}^{+\{h\}}$

The differences between the basic and nonbasic notations reflect the differing roles of the basic and nonbasic variables in a basic solution

For x_{Nj} the breakpoint γ_{Nj} is fixed, and the notation 'centers' the slopes and other breakpoints around it. For x_{Bi}. the notation centers the breakpoints and slopes around the fixed slope x_{Bi}.

5. **Statement and proof of a piecewise-linear simplex algorithm**

Given a basic feasible solution from which to start, this algorithm finds a basic optimal solution (if a finite minimum exists) provided that the number of breakpoints is finite and that all basic feasible solutions are non-degenerate . The proof of the algorithm adopts the familiar arguments of the linear case. First it is shown that the algorithm eventually stops. Then it is proved that, when the algorithm does stop, it has either found a finite optimal solution or has demonstrated that none exists. The $P-L$ conjugacy properties make possible a concise proof of optimality. Under the further hypothesis that basic feasible solutions exist whenever the constraints are feasible , the optimality proof simultaneously establishes a strong duality theorem for piecewise-linear programming.

5.1. Statement of the algorithm

The piecewise-linear simplex algorithm starts with some basic feasible solution $\bar{x}$ defined by a basis matrix B and a vector of breakpoint values γ_N:

$$\bar{x}_{Nj} = \gamma_{N,} \qquad B\bar{x}_B = b - \sum_j a_{Nj,}\gamma_{Nj}$$

A vector C_B of finite slopes is chosen to correspond to the values $\bar{x}_B$ of the basic variables:

$C_{Bi}=c_{Bi}^{h}$ if $\gamma_{Bi}^{h} < \bar{x}_{Bi}\gamma_{Bi}^{(h+1)}$

An iteration of the algorithm then proceeds as follows:

(1) Solve $\bar{\pi}B = C_B$.

(2) Test for optimality: If the no basic variables satisfy

$C_{Nj}^{-\{1\}} \leq \bar{\pi}a_{Nj} \leq C_{Nj}^{+\{1\}}$ for all j

then the basic solution is optimal. STOP.

(3) Select entering variable: Choose a variable x_{NP} such that

$d_{NP}^{-} = C_{NP}^{-\{1\}} - \bar{\pi}a_{Np} > 0,$ or

$d_{NP}^{+} = C_{NP}^{+\{1\}} - \bar{\pi}a_{Np} < 0,$

(4) Solve

$By_B = -a_{Np}$ if $d_{NP}^{-} > 0$, or

$By_B = a_{Np}$ if $d_{NP}^{+} < 0$

(5) Test for unboundedness: If

$\gamma_{Bq}^{+\{1\}} = +\infty$ for all $y_{Bi} < 0$

$\gamma_{Bq}^{-\{1\}} = -\infty$ for all $y_{Bi} > 0$

$\gamma_{Np}^{+\{1\}} = +\infty$ if $d_{NP}^{+} < 0$

$\gamma_{Np}^{-\{1\}} = -\infty$ if $d_{NP}^{-} > 0$

then the objective can decrease without bound. STOP.

(6) Select leaving variable: Choose any variable x_{Bq} and breakpoint $\gamma_{Bq}^{+\{r_q\}}$ or $\gamma_{Bq}^{-\{r_q\}}$ such that

(a) $(x_{Bq} - \gamma_{Bq}^{+\{r_q\}})/y_{Bq} = \bar{\theta}$ and $y_{Bq} < 0,$ or

(b) $(x_{Bq} - \gamma_{Bq}^{-\{r_q\}})/y_{Bq} = \bar{\theta}$ and $y_{Bq} > 0$

or choose any breakpoint $\gamma_{Bq}^{+\{r\}}$ or $\gamma_{Bq}^{-\{r\}}$ such that

(C) $-(\gamma_{Np} - \gamma_{Np}^{+\{r\}} = \bar{\theta}$ and $d_{NP}^{+} < 0,$ or

(d) $(\gamma_{Np} - \gamma_{Np}^{\{r\}} = \bar{\theta}$ and $d_{NP}^{-} > 0$

provided $\bar{\theta}$ is small enough that $d_{NP}^{+} < 0$ and

$d_{NP}^{+}(\bar{\theta}) = c_{Np}^{+\{r\}} - \sum_{y_{Bi<0}} c_{Bi}^{+\{ri-1\}} y_{Bi} - \sum_{y_{Bi>0}} c_{Bi}^{\{ri-1\}} y_{Bi} < 0,$

or $\bar{\theta}$ is small enough that d $d_{NP}^{+} > 0$ and

$d_{NP}^{-}(\bar{\theta}) = c_{Np}^{-\{r\}} - \sum_{y_{Bi<0}} c_{Bi}^{+\{ri-1\}} y_{Bi} - \sum_{y_{Bi>0}} c_{Bi}^{\{ri-1\}} y_{Bi} > 0$

where the breakpoints r_i and r are defined by

$(x_{Bi} - \gamma_{Bi}^{+\{r_i-1\}})/y_{Bi} < \bar{\theta} \leq (x_{Bi} - \gamma_{Bi}^{+\{r_i\}})/y_{Bi}$ for $y_{Bi} < 0,$

$(x_{Bi} - \gamma_{Bi}^{\{r_i-1\}})/y_{Bi} < \bar{\theta} \leq (x_{Bi} - \gamma_{Bi}^{\{r_i\}})/y_{Bi}$ for $y_{Bi} > 0,$

$-(\gamma_{Np} - \gamma_{Np}^{+\{r_i-1\}})/y_{Bi} < \bar{\theta} \leq (\gamma_{Np} - \gamma_{Np}^{+\{\ r\}})/y_{Bi}$ for$d_{NP}^{+} < 0$,,

$(\gamma_{Np} - \gamma_{Np}^{\{r_i-1\}})/y_{Bi} < \bar{\theta} \leq (\gamma_{Np} - \gamma_{Np}^{\{r\ \}})/y_{Bi}$ for $d_{NP}^{-} > 0,$

(7) Update basic solution: Reset

$\bar{x}_B \leftarrow \bar{x}_B - \overline{\theta y}_B,$

$\bar{x}_{Np} \leftarrow \quad \gamma_{NP}^{+} + \bar{\theta}$, if $d^{+}_{Np} < 0$,

$\bar{x}_{Np} \leftarrow \quad \gamma_{NP}^{+} - \bar{\theta}$, if $d^{-}_{Np} > 0$, and set C_{Bi} accordingly for i≠q

$C_{Bi} = c_{Bi}^{+\{ri-1\}}$ for $y_{Bi} < 0,$

$C_{Bi} = c_{Bi}^{+\{ri-1\}}$ for $y_{Bi} > 0,$

(8) Change basis, according to the choice made in step (6):

If (a) or (b), replace a_{Bq} by a_{Np} in$B$.

Set $C_{Np} = C_{NP}^{+\{r\}}$ (if $d_{NP}^{+} < 0$) or $C_{Np} = C_{NP}^{-\{r\}}$ (if $d_{NP}^{-} > 0$)

Set $\gamma_{Bq} = \gamma_{Bq}^{+\{rq\}}$ (if $Y_{\mathrm{Bq}} < 0$ or $\gamma_{Bq} = \gamma_{Bq}^{-\{rq\}}$ (if $Y_{\mathrm{Bq}} > 0$).

If (c) or (d), set $\gamma_{Np} = \gamma_{Nq}^{+\{r\}}$ (if $d_{NP}^{+} < 0$) or $\gamma_{NP} = \gamma_{NP}^{-\{r\}}$ (if $d_{NP}^{-} > 0$)

The piecewise-linear simplex algorithm carries out a series of these iterations,

Each starting from the B, C_B, γ_N and $\bar{x}$ determined at the end of the preceding iteration. The algorithm stops when, at some iteration, the termination conditions of step (2) or step (5) are satisfied.

CONCLUSION

The first develops and justifies a piecewise-linear simplex algorithm,that each piecewise-linear term has a finite number of pieces, that a basic feasible starting basis can be found, and that all bases are non-degenerate. Then second shows how the algorithm may be extended to permit the relaxation of these assumptions. The third carries out the computational analysis, and concludes with a summary of applications of piecewise-linear programming. The first begins with a survey of the origins of the subject in second. The bounded-variable simplex algorithm is then introduced in third , to provide a point of comparison and to establish simplex terminology.

The develop the piecewise-linear algorithm. Introduce convex separable piecewise-linear functions and piecewise-linear programs, respectively, and state their elementary properties. The defines basic solutions for piecewise-linear programs, and derives the essential operations of moving from one basic solution to another. Finally, presents a detailed statement and proof of the algorithm.

REFERENCE:

1. Premoli, “Piecewise linear programming: The compact (CPLP) algorithm”, Technical report, Gruppo Sistemi e Circuiti, Istituto Elletrotecnico Nazionale Galileo Ferraris (Torino, Italy, 1982).

2. R.T. Rockafellar, “Convex programming and systems of elementary monotonic relations”,*Journal of Mathematical Analysis and Applications* 19 (1967) 543–564.

3. R.T. Rockafellar, “Monotropic programming: Descent algorithms and duality”, in: O. Mangasarian et al., eds., *Nonlinear programming 4* (Academic Press, New York, 1981) pp. 327–365.

4. R.T. Rockafellar,*Network flows and monotropic optimization* (Wiley-Interscience, New York, 1984).

5. R.D. Snyder, “Programming with piecewise-linear objective functions”, Working paper 9/81, Department of Econometrics and Operatibns Research, Monash University (Clayton, Victoria, Australia, 1981).

6. R.D. Snyder, “Linear programming with special ordered sets”,*Journal of the Operational Research Society* 35 (1984) 69–74.

7. Bitran, G. R., & Novaes, A. J. 1973. Linear programming with a fractional objective function. Journal of Operations Research, 21, 22-29.

8. Charnes, A., & Cooper, W. W. 1962. Programming with linear functional. Naval Research Logistics Quarterly, 9, 181-186.

9. R. Fourer, “A simplex algorithm for piecewise-linear programming III: computational analyse an application, “ Technical report 86- 03,Department of Industrial Engineering and Management Science, Northwestern University (Evanston , IL, 1986)

10. Dantzig, G. B. 1962. Linear Programming and extension. Princeton University Press, Princeton, NewJersey

11. Odior, A. O. 2012. An approach for solving linear fractional programming problems. International Journal of Engineering and Technology, 1, 298-304

12. FOURER R., A Simplex algorithm for piecewise-linear programming I: Derivation and proof, Mathematical Programming 33, pp 204-233, 1985.

13. FOURER R., A Simplex algorithm for piecewise-linear programming II: Finiteness, feasibility and degeneracy, Mathematical Programming 41,pp 281-315, 1988.

14. FOURER, R., A Simplex algorithm for piecewise-linear programming III: computational analysis and applications, Mathematical Programming 53, 1992, pp 213-235.

15. FOURER R. and R.E. MARSTEN, Solving piecewise-linear programs: experiments with a simplex approach, ORSA Journal on Computing 4,1992, pp 16-31.