A Study of Bifurcation Theory and its Application in Differential Equations

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Abstract: **Bifurcation theory is a subject with classical mathematical origin. The modern development of this subject dates back to the pioneering work of Poincare, and over the past four decades this theory has witnessed rapid developments with new ideas and methods from dynamical systems theory, singularity theory and many others. In general, in a dynamical system, a parameter is allowed to vary, and then the differential system may change. Equilibrium can become unstable and a periodic solution may appear or a new stable equilibrium may appear making the previous equilibrium unstable. The value of parameter at which these changes occur is known as "bifurcation value" and the parameter that is varied is known as the "bifurcation parameter". In recent years many types of bifurcations of flow and maps have been studied and classified including saddle node, Hopf, umbilic, zip, Homo clinic tangencies, period doubling and cusp bifurcations. It is our belief that in the years to come the bifurcation theory plays a more active role in various application domains of Science and Technology. In the first section several types of bifurcations have shown through examples. In the second section the structural stability is introduced and studied. The most important two bifurcations have dealt with in detail in the next two sections, where also sufficient conditions are given for these bifurcations. Applications of bifurcation will be given in fifth section.**

***Index Terms* -** Bifurcation, Equilibrium, structural stability, saddle node bifurcation, Hopf bifurcation.

I. BIFURCATION

1.1. Introduction

Bifurcation theory is a subject with classical mathematical origin. The modern this subject dates back to the pioneering work of Poincaré, and over the past four decades this theory has witnessed rapid developments with new ideas and methods from dynamical systems theory, singularity theory and many others.

Bifurcation theory attempts to explain various phenomena that have been discovered and described in the natural sciences over the centuries. The buckling of the Euler rod, the appearance of Taylor vortices, and the onset of oscillations in an electric circuit, for instance, all have a common cause: A specific physical parameter crosses a threshold, and that event forces the system to the organization of a new state that differs considerably from that observed before. Mathematically speaking, the following occurs: The observed states of a system correspond to solutions of nonlinear equations that model the physical system. A state can be observed if it is stable, an intuitive notion that is made precise for a mathematical solution.

One expects that a slight change of a parameter in a system should not have a big influence, but rather that stable solutions change continuously in a unique way. That expectation is verified by the Implicit Function Theorem. Consequently, as long as a continuous branch of solutions preserves its stability, no dramatic change is observed when the parameter is varied. However, if that "ground state" loses its stability when the parameter reaches a critical value, then the state is no longer observed, and the system itself organizes a new stable state that "bifurcates" from the ground state. Bifurcation is a paradigm for non-uniqueness in Nonlinear Analysis. Bifurcation Theory provides the mathematical existence of bifurcation scenarios observed in various systems and experiments. A necessary condition is obviously the failure of the Implicit Function Theorem.

The bifurcation is the qualitative change in the phase portrait. The bifurcation occurs at those parameter values for which the phase portrait is not topologically equivalent to those belonging to nearby parameter values. This is formulated in the following definition. Consider the equation $\dot{x}(t) = f(x(t), \lambda)$, where $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is a continuously differentiable function and $\lambda \in \mathbb{R}^k$ is a parameter.

Definition.

The parameter value $\lambda_0 \in \mathbb{R}^k$ is called regular, if there exists $\delta > 0$, for which $|\lambda - \lambda_0| < \delta$ implies that the system $f(\cdot, \lambda)$ is topologically equivalent to the system $f(\cdot, \lambda_0)$. At the parameter value $\lambda_0 \in \mathbb{R}^k$ there is a bifurcation if it is not regular.

In general, in a dynamical system, a parameter is allowed to vary, and then the differential system may change. Equilibrium can become unstable and a periodic solution may appear or a new stable equilibrium may appear making the previous equilibrium unstable. The value of parameter at which these changes occur is known as "bifurcation value" and the parameter that is varied is known as the "bifurcation parameter".

In the first section several types of bifurcations will be shown through examples. In the second section the structural stability is introduced and studied. The most important two bifurcations will be dealt with in detail in the next two sections, where also sufficient conditions will be given for these bifurcations. Applications of bifurcation will be given in fifth section.

1.2. Basic concepts of bifurcation analysis

As it is stated above, in dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden "qualitative" or topological change in its behavior. Generally, at a bifurcation, the local stability properties of equilibrium, periodic orbits or other invariant sets changes. It has two types;

Local bifurcations, which can be analyzed entirely through changes in the local stability properties of equilibrium, periodic orbits or other invariant sets as parameters cross through critical thresholds; and

Global bifurcations, which often occur when larger invariant sets of the system "collide" with each other, or with equilibrium of the system. They cannot be detected purely by a stability analysis of the equilibrium.

Equilibrium points

In dynamical systems, only the solutions of linear systems may be found explicitly. The problem is that in general real life problems may only be modeled by nonlinear systems. The main idea is to approximate a non linear system by a linear one (around the equilibrium point). Of course, we do hope that the behavior of the solutions of the linear system will be the same as the nonlinear one. But this is not always true. Before the linear stability analysis, we give some basic definitions below.

Definition (Equilibrium Point):

Consider a nonlinear differential equation $x'(t) = f(x(t), u(t))$, where f is a function mapping $\mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. A point $\bar{x}$ is called an equilibriumpoint if there is a specific $\bar{u} \in R^m$ such that

$$f(x(t), u(t)) = 0$$

Suppose $\bar{x}$ is an equilibrium point (with the input $\bar{u}$). Consider the initial condition x (0) = $\bar{x}$, and applying the input $u(t) = \bar{u}$ for all $t \geq t_0$, then resultingsolution $x(t)$ satisfies $x(t) = \bar{x}$, for all $t \geq t_0$. That is why it is called an equilibriumpoint or solution.

Example:

As an example, consider the logistic growth equation (the rate of population density) $x' = rx(1 - \frac{x}{K})$, where $x(t)$ denotes the population density at time t, r and K are positive constants, K is the carrying capacity. Then by setting right hand side function equal to zero, $f(x) = rx(1 - \frac{x}{K}) = 0$, we obtain two equilibriumpoints $x = 0$ and$x = K$. **1.3. Normal forms of Elementary bifurcations**

In this section, we discuss several types of bifurcations, saddle node, Trans critical, pitchfork and Hopf bifurcation. The first three types of bifurcation occur in scalar and in systems of differential equations. The fourth type called Hopf bifurcation does not occur in scalar differential equations because this type of bifurcation involves a change to a periodic solution. Scalar autonomous differential equations cannot have periodic solutions. Hopf bifurcation occurs in systems of differential equations consisting of two or more equations. This type is also referred to as a "Poincare-Andronov-Hopf bifurcation".

1.3.1. Phase portrait

Consider the differential equation $\dot{x} = \lambda - x$ in which $\lambda \epsilon \mathbb{R}$ is a parameter.For a given value of λ the equilibrium is the point $x = \lambda$ .This point is globally asymptotically stable for all values of λ, that is trajectories are tending to this point.The phase portrait for different values of λ can be shown in a coordinate system, wherethe horizontal axis is for λ and for a given value of λ the corresponding phase portraitis given on the vertical line atλ, as it is shown in Figure 1.1. This Figure shows that the phase portrait is the same for all values ofλ, that is all values of λ are regular, i.e. there is no bifurcation. The topological equivalence of the phase portraits corresponding to different values of λcan be formally verified by determining the homeomorphism the orbits to each other. For example, the orbits for λ= 0 can be taken to those belongingtoλ= 1 by the homeomorphism $h(p) = p - 1$.

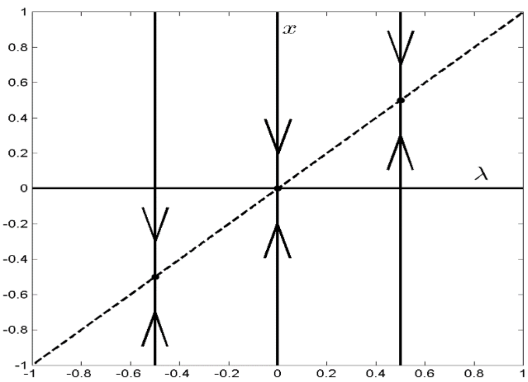


Figure 1.1. The phase portrait of the differential equation $\dot{x} = \lambda - x$ for different values of the parameter λ.

1.3.2. Saddle-node bifurcation

A saddle-node bifurcation or tangent bifurcation is a collision and disappearance of two equilibrium in dynamical systems. In autonomous systems, this occurs when the critical equilibrium has one zero eigenvalue. This phenomenon is also called fold or limit point bifurcation.

Example 1.3.2.

Consider the differential equation$\dot{x} = \dot{\lambda} - x^2$, where $\lambda \epsilon \mathbb{R}$ is a parameter. In this case the existence of the equilibrium depends on the parameterλ. If $\lambda < 0$, then there is no equilibrium, for $\lambda = 0$ the origin $x = 0$ is an equilibrium, and for $\lambda > 0$ there are two equilibria $x = \pm\sqrt{\lambda}$.The phaseportrait can be shown for different values of $\lambda$by using the same method as in the previousexample, as it is shown in Figure 1.3.2. It can be seen in the Figure that the bifurcation is atλ = 0, since the phase portrait is different for positive and negative values of the parameter. The values $\lambda \neq 0$ are regular, because choosing a positive or negative valueof $\lambda$the phase portrait does not change as λ is varied in a suitably small neighborhood.The topological equivalence of the phase portraits corresponding to different non-zero values of λ can be formally verified by determining the homeomorphism taking the orbits to each other. For example, the orbits for λ< 0 can be taken to each other by the homeomorphism $h(p) = p$. For positive values of λ the orbits for can be taken to each other by a piece-wise linear homeomorphism. The bifurcation in this example is called fold or saddle-node bifurcation.

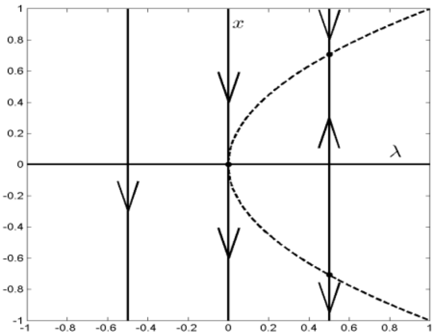


Figure 1.2: Fold or saddle-node bifurcation in the differential equation $\dot{x} = \lambda - x^2$ at $\lambda = 0$

1.3.3. Trans critical bifurcation

In a Trans critical bifurcation, two families of fixed points collide and exchange their stability properties. The family that was stable before the bifurcation is unstable after it. The other fixed point goes from being unstable to being stable.

Example 1.3.3.

Consider the differential equation $\dot{x} = \lambda x - x^2$, where $\lambda \epsilon \mathbb{R}$ is a parameter. The point x = 0 is an equilibrium for any value of λ.Besides this point x = λ is also an equilibrium, therefore in the case $\lambda \neq 0$ there aretwo equilibria, while for $\lambda = 0$ there is only one. Hence there is a bifurcation at $\lambda = 0$.The phase portrait can be shown for different values of λ by using the same method asin Example 1.1, as it is shown in Figure 1.3. It can be seen in the Figure that the bifurcation is at $\lambda = 0$, since the phase portrait for non-zero values of the parameter is different from that of belonging to $\lambda = 0$. The values $\lambda \neq 0$ are regular, because choosinga positive or negative value of λ the phase portrait does not change as λ is varied in a suitably small neighbourhood. For negative values of λthe point x = 0 is stable and $x = \lambda$ is unstable, while for positive λ values it is the other way around. This bifurcation iscalled trans critical because of the exchange of stability. The topological equivalence of thephase portraits corresponding to different non-zero values of λ can be formally verified by determining the homeomorphism taking the orbits to each other.

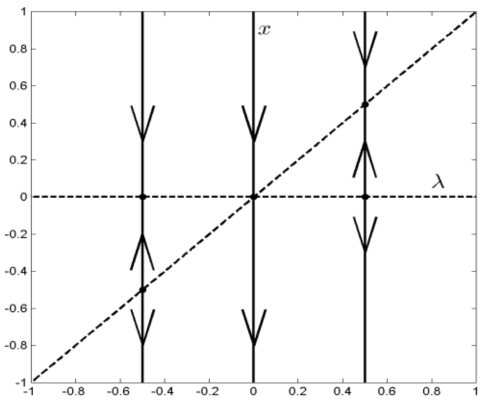
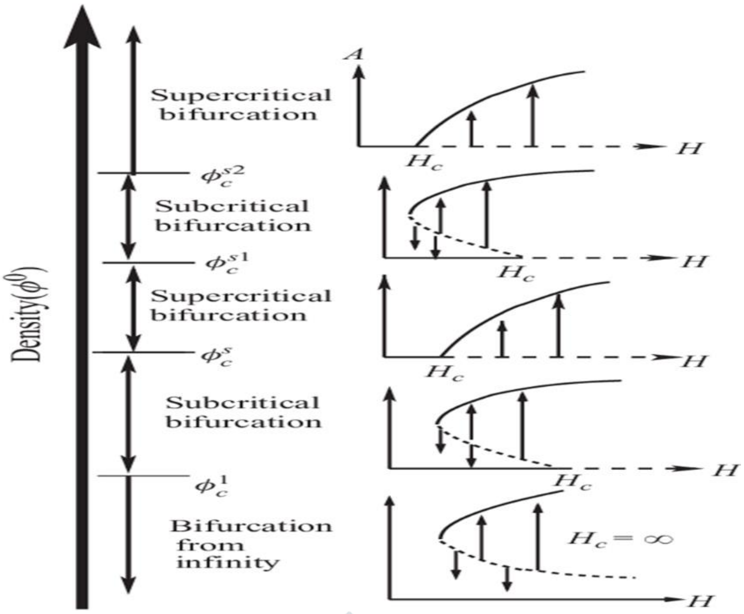


Figure 1.3: Trans critical bifurcation in the differential equation $\dot{x} = \lambda x - x^2$ at $\lambda = 0$

1.3.4. The pitchfork bifurcation

In pitchfork bifurcation one family of fixed points transfers its stability properties to two families after or before the bifurcation point. If this occurs after the bifurcation point, then pitchfork bifurcation is called supercritical. Similarly, a pitchfork bifurcation is called subcritical if the nontrivial fixed points occur for values of the parameter lower than the bifurcation value. In other words, the cases in which the emerging nontrivial equilibria are stable are called supercritical whereas the cases in which these equilibria are called subcritical.



Example.1.3.4.

Consider the differential equation $\dot{x} = \lambda x - x^3$, where $\lambda \epsilon \mathbb{R}$ is a parameter. The point x = 0 is an equilibrium for any value of λ.Besides this point $x = \pm\lambda$ is also an equilibriumif $\lambda > 0$ .Thus for $\lambda < 0$ there is a unique equilibrium, while for $\lambda > 0$, there are 3 equilibria. Hence there is a bifurcation at $\lambda = 0$. The phase portrait can be shown for different values of λ by using the same method as in Example 1.3.1, as it is shown in Figure 1.4. It can be seen in the Figure that the bifurcation is at λ = 0, since the phase portrait for non-zero values of the parameter is different from that of belonging to $\lambda = 0$. The values $\lambda \neq 0$ are regular, because choosing a positive or negative value of λ the phase portrait does not change as λ is varied in a suitably small neighbourhood. For negative values of λ the equilibrium point $x = 0$ is globally stable. For positive values of λ the points $x = \pm\sqrt{\lambda}$ take over stability. This bifurcation is called pitchfork bifurcation because of the shape of the bifurcation curve. The topological equivalence of the phase portraits corresponding to different non-zero values of λ can be formally verified by determining the homeomorphism taking the orbits to each other.

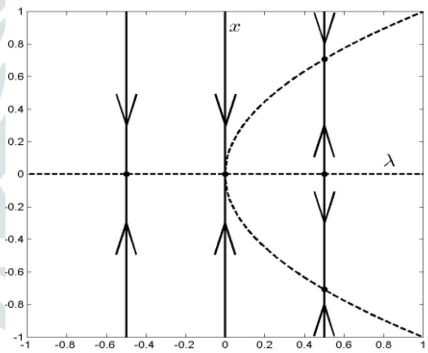


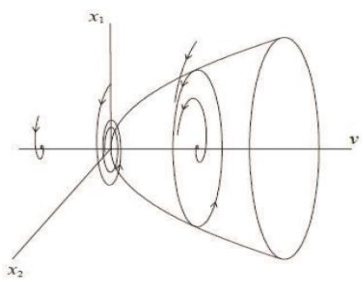
Figure 1.4: Pitchfork bifurcation in the differential equation $\dot{x} = \lambda x - x^3$, at $\lambda = 0$.

Bifurcation in two dimension

1.3.5. Hopf bifurcation

Definition:

A Hopf or Poincare-Andronov-Hopf bifurcation is a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of linearization around the fixed point cross the imaginary axis of the complex plane.



Bifurcation diagram corresponding to Supercritical Hopf bifurcation

Example 1.3.5 (Andronov–Hopf bifurcation).

Consider the differential equation $\dot{r} = \lambda r + \sigma r^3$, $\dot{}\ \varphi = 1$ given in polar coordinates, where $\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ are parameters.First, fix the value $\sigma = -1$ (any $\sigma < 0$ yields the same phenomenon) and see how thephase portrait changes as the value of λ is varied. The origin is an equilibrium for any value of λ and its stability can be easily determined from the differential equation for r. Namely, in the case $\lambda < 0$, we have $\dot{r} = \lambda r - r^3 < 0$, hence r is strictly decreasing and converges to zero, therefore the solutions tend to the origin. However, for $\lambda > 0$ and $r < \sqrt{\lambda}$ we have $\dot{r} = r(\lambda r - r^2) > 0$, hence r is strictly increasing, therefore the originis unstable. Moreover, for $r = \sqrt{\lambda}$ we have$\dot{r} = 0$, that is the circle with radius $\sqrt{\lambda}$ is a periodic orbit that is orbitally asymptotically stable, because inside the circle $\dot{r} > 0$ and outside $\dot{r} < 0$. This phenomenon is illustrated in Figure 1.6. In the Figure the behaviour of r is shown as λ is varied. The bifurcation is at $\lambda = 0$ and the values $\lambda \neq 0$ are regular.

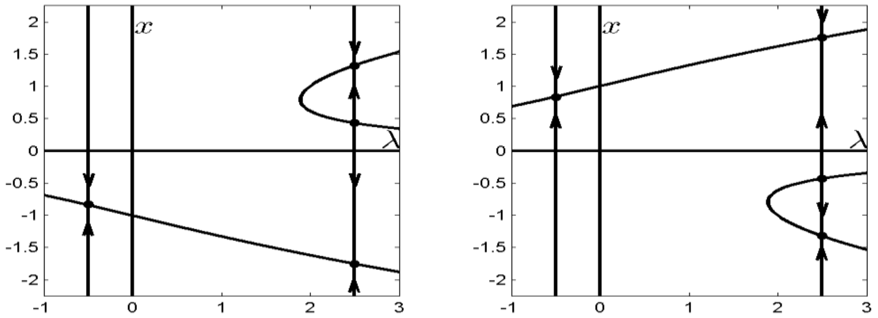


Figure 1.5: The bifurcation diagram with respect to λin the differential equation$\dot{r} = \lambda r + \sigma r^3$ for $\mathcal{E} = -1$ and for $\mathcal{E} = 1$.

The bifurcation in the two dimensional phase space is shown in Figure 1.7. If $\lambda < 0$, then the origin is globally asymptotically stable, while for λ >0 the origin is unstable and the stability is taken over by a stable limit cycle, the size of which is increasing as $\sqrt{\lambda}$. This bifurcation is called supercritical Andronov–Hopf bifurcation. Returning to the differential equation $\dot{r} = \lambda r + \sigma r^3$, $\dot{\varphi} = 1$ consider the case of positive σ values, say let $\sigma = 1$. The origin is an equilibrium again the stability of which is changed in the same way with λ as before, however, the periodic solution now appears for $\lambda < 0$, and it is unstable. The origin loses its stability for $\lambda > 0$, however, in this case the periodic orbit does not take over the stability, the trajectories tend to infinity. If $\lambda < 0$, then the origin is stable but its domain of attraction is only the interior of the periodic orbit. If $\lambda > 0$, then the origin is unstable and the trajectories tend to infinity. This bifurcation is called subcritical Andronov–Hopf bifurcation.

In the previous examples the bifurcation occurred locally in the phase space, in a neighbourhood of an equilibrium. These kind of bifurcations are called local bifurcations.

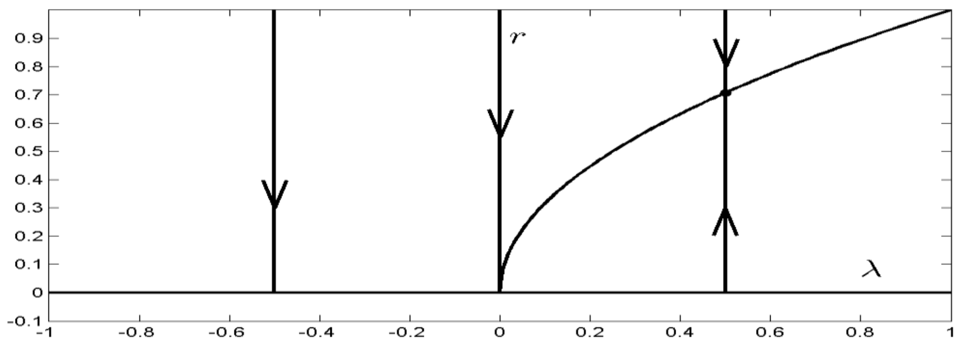


Figure 1.6: Bifurcation of the differential equation $\dot{r} = \lambda r - r^3$at $\lambda = 0$.

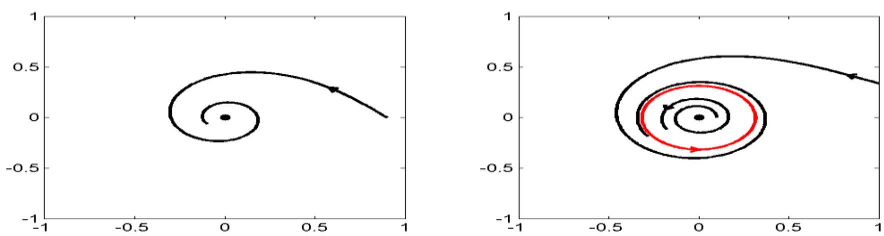


Figure 1.7: Supercritical Andronov-Hopf bifurcation, the origin looses its stability and a stable limit cycle is born.

1.4 Necessary conditions of bifurcations

The examples of the previous section show that local bifurcation may occur at non hyperbolic equilibria. This statement will be proved generally in this section. The notion of local bifurcation is defined first. Consider a general system of the form $\dot{x}(t) = f(x(t), \lambda)$, where$f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$is a continuously differentiable function, and $\lambda \in \mathbb{R}^k$ is a parameter.

Definition

The pair (x_0, λ_0) is called locally regular, if there exist a neighbourhood $U \subset \mathbb{R}^n$of x_0 and $\delta > 0$, such that for $|\lambda - \lambda_0| < \delta$ the systems and$^f/_U(\cdot, \lambda_0)$ and $^f/_U(\cdot, \lambda)$are topologically equivalent (that is the phase portraits are topologically equivalent in U). There is a local bifurcation at (x_0, λ_0), if $(x_0, \lambda_0)$is not locally regular.

Proposition 1.4.1.

If $f(x_0, \lambda_0), \neq 0$,then (x_0, λ_0), is locally regular.

Proof.

For simplicity, the proof is shown for the case $n = 1$. Without loss of generality one can assume that $f(x_0, \lambda_0) > 0$. Then the continuity of f implies that there exist a neighbourhood $U \subset \mathbb{R}^n$of $x_0$and $\delta > 0$, such that in the set $\bar{U} \times [\lambda_0 - \delta, \lambda_0 + \delta]$ the value of f is positive. Hence in this set the trajectories are segments directed upward, as it is shown in Figure 1.8. Hence the phase portraits are obviously topologically equivalent in U for all$\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. (The homeomorphism taking the orbits into each other is the identity.)

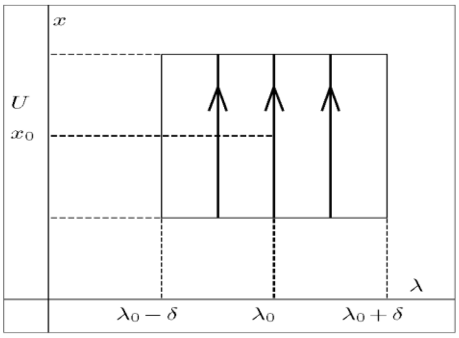


Figure 1.8: The trajectories in a neighbourhood U of a non-equilibrium point are the same for all values of the parameter λ.

Proposition 1.4.2.

If $f(x_0, \lambda_0) = 0$ and $\partial_x f(x_0, \lambda_0)$is hyperbolic, then $(x_0, \lambda_0)$is locally regular. (Here $\partial_x f(x_0, \lambda_0)$ denotes the Jacobian matrix of f.)

Proof.

For simplicity, the proof is shown again for the case n = 1. Without loss of generality one can assume that $\partial_x f(x_0, \lambda_0) < 0$. Then according to the implicit function theorem there exist $\delta > 0$ and a differentiable function $g : (\lambda_0 - \delta, \lambda_0 + \delta) \to \mathbb{R}$, for which g($\lambda_0$) = x_0 and f($g(\lambda), \lambda) \equiv 0$ for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, moreover, there is neighbourhood U of x_0, such that in other points of the set $U \times (\lambda_0 - \delta, \lambda_0 + \delta)$ thefunction f is nonzero. Since f is continuously differentiable, the number δ can be chosen so small that $\partial_x f(g(\lambda), \lambda) < 0$ holds for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Hence for these values of λ there is exactly one stable equilibrium in U and the trajectories tend to this point as it is shown in Figure 1.9. Therefore the phase portraits are obviously topologically equivalent in U for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. (The homeomorphism taking the orbits into each other is a translation taking the steady states to each other).

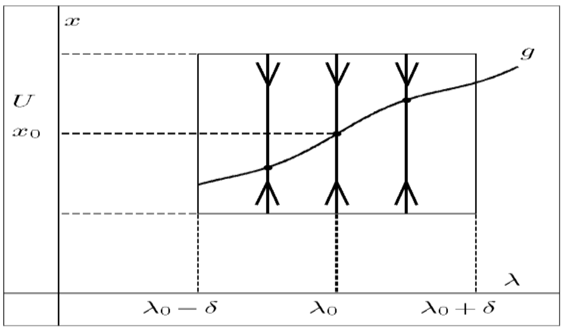


Figure 1.9: The phase portraits in a neighbourhood U of a hyperbolic equilibrium are the same for all values of the parameter λ.

II. Stability

2. 1. Structural stability

In the course of studying bifurcations, a system of differential equations $\dot{x}(t) = f(x(t))$ was considered to be a member of a k parameter family $\dot{x}(t) = f(x(t), \lambda)$ and it was investigated how the phase portrait is changing as the k dimensional parameter λ is varied. Here a more general approach is shown, where all C^1 perturbations are investigated together with the system $\dot{x}(t) = f(x(t))$, that is the right hand side f is considered as an element of a function space. In the case of bifurcations a value of the parameter was called regular if the corresponding system was topologically equivalent to all other systems belonging to nearby parameter values. The generalization of this is the structurally stable system that is topologically equivalent to all other systems that are sufficiently close in the C^1 norm.

For formulating the definition in abstract terms let X be a topological space and let $\sim\subset X \times X$ be an equivalence relation. In our case the topological space will be a suitable function space with the C^1 topology and the equivalence relation will be the topological equivalence.

Definition 2.1.1

An element $x \in X$ is called structurally stable, if it has a neighbourhood $U \subset X$, for which $y \in U$ implies $x \sim y$.

An element $x \in X$ is called a bifurcation point, if it is not structurally stable.

In other words, we can say that $x \in X$ is structurally stable, if it is an interior point of an equivalence class, and it is bifurcation point if it is a boundary point of an equivalence class. This interpretation enables us to define the co-dimension of a bifurcation. The co-dimension of a bifurcation is the co-dimension of the surface that forms the boundary in a neighbourhood of the given bifurcation point.

In Figure 2.1 the point A is structurally stable, point B is a one co-dimensional bifurcation point and point C is a two co-dimensional bifurcation point. This can also be formulated as follows. There is a curve through B that intersects both domains that are separated by the border containing B, while there is no such curve through C. The classes touching C can be reached by a two parameter family, i.e. a surface in the space. This is formulated rigorously in the following definition.

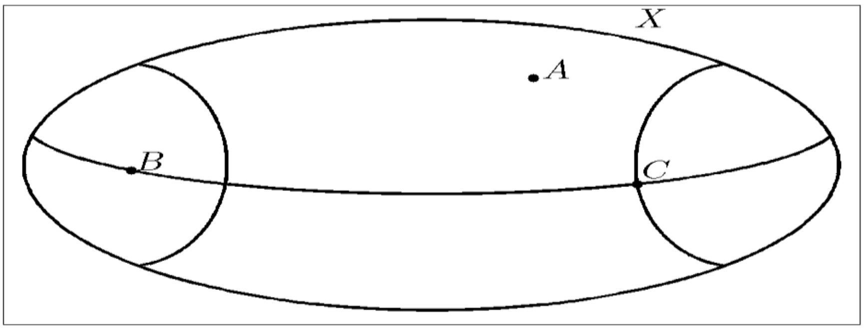


Figure 2.1: Structurally stable (A), one co-dimensional (B) and two co-dimensional (C) bifurcation points in a topological space X.

Definition 2.1.2

A bifurcation point $x \in X$ is called k co-dimensional, if there is a continuous function $g : \mathbb{R}^k \to X$, for which $g(0) = x$ and the point x has a neighbourhood U and it has an open dense subset V , such that for all $y \in V$ there is an $\alpha \in \mathbb{R}^k$ satisfying $g(\alpha) \sim y$, and k is the smallest dimension with these properties.

2.2. Structural stability of one dimensional system

Let us introduce the space $X = C^1(S^1, \mathbb{R})$ that consists of continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$, which are periodic with period 1, that is $f(x + 1) = f(x)$ for all$x \in \mathbb{R}$. This space will be endowed with the norm

$$\|f\|_1 = \max_{[0,1]}|f| + \max_{[0,1]}|f'|$$

It will be shown that those systems are structurally stable, for which all equilibria are hyperbolic. Introduce the following notation for these systems.

$$G = \{f \in X : f(x) = 0 \Rightarrow f'(x) \neq 0\}$$

In the proof of the theorem the following notion and lemma are crucial.

Definition 2.2.1

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. The value y is called a regular value of f, if $f(x) = y$implies $f'(x) \neq 0$. In the higher dimensional case when $f : \mathbb{R}^n \to \mathbb{R}^n$the assumption is that $f(x) = y$ implies $det f'(x) \neq 0$.If y is not a regular value, then it is called a critical value of f.

Lemma 2.2.1 (Sard).

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function, then the set of its critical values has measure zero.

Using the lemma the following proposition can be proved.

Proposition 2.2.1.

The above set G of dynamical systems having only hyperbolic equilibria is dense in the space $X = C^1(S^1, \mathbb{R})$.

Proof.

A function is in the set G, if and only if 0 is its regular value. Let $f \in X$ and $\varepsilon > 0$ be arbitrary. It has to be shown that there exists g ∈ G, for which $\|f - g\|_1 < \varepsilon$. If 0 is a regular value of f, then $g = f$ is a suitable choice, since then $f \in G$. If 0 is not a regular value, then chose a positive regular value $c < \varepsilon$. The existence of this c is guaranteed by Sard's lemma. Then let $g = f - c$, therefore $\|f - g\|_1 = c < \varepsilon$and $g(x) = 0$ implies $f(x) = c$, hence the regularity of c yields $f'(x) \neq 0$ which directly gives $g'(x) \neq 0$. Thus 0 is a regular value of g, that is $g \in G$.

Proposition 2.2.2

Let $f \in X$ and assume that for some $x \in (0,1)$ we have $f(x) = 0 = f'(x)$. Then for any $\varepsilon > 0$ and $\alpha > 0$ there exists a function $g \in X$, for which the following statements hold.

1. $f(y) = g(y)$ for all $y \notin (x - \alpha, x + \alpha)$,

2. g is constant 0 in a neighbourhood of x,
3. $\|f - g\|_1 < \varepsilon$.

Proof.

Let $\eta : \mathbb{R} \to [0,1]$ be a C^1 function (in fact it can be chosen as a C^∞ function), that is constant zero outside the interval $[-1,1]$ and constant 1 in the interval $(-\frac{1}{2},\frac{1}{2})$.The maximum of $|\eta'|$ is denoted by M. The assumption on f implies that there exists a positive number $\beta < \alpha$, for which

$$|f(y)| < \frac{\varepsilon}{4M}|y-x| \tag{2.1}$$

(2.1) holds for all $y \in (x-\beta, x+\beta)$. Let $\delta < \beta$ be a positive number, for whichmax

$$\max_{[x-\delta,x+\delta]} |f| < \frac{\varepsilon}{2}$$

and

$$\max_{[x-\delta,x+\delta]} |f'| < \frac{\varepsilon}{4} \tag{2.2}$$

Then let $g \in X$ be given as follows $(y) = f(y)\left(1-\eta\left(\frac{y-x}{\delta}\right)\right)$.

Now it is checked that g satisfies the conditions.

If $|y-x| \geq \alpha$, then $|y-x| > \delta$yielding $\eta\left(\frac{y-x}{\delta}\right) = 0$, hence the first condition holds, i.e. $g(y) = f(y)$.

$If\ |y-x| < \delta/2$, then $\eta((y-x)/\delta) = 1$, hence the second condition holds, i.e. g is constant zero in a neighbourhoodof x. In order to check the last condition we use that

$$f(y) - g(y) = f(y)\eta\left(\frac{y-x}{\delta}\right) \tag{2.3}$$

Therefore $f-g$ is zero in $[x-\delta, x+\delta]$, hence it is enough to prove that for all$y \in [x-\delta, x+\delta]$ the following holds$|f(y) - g(y)| < \frac{\varepsilon}{2}$

and

$$|f'(y) - g'(y)| < \frac{\varepsilon}{2} \tag{2.4}$$

This yields for $y \in [x-\delta, x+\delta]$ that $|f(y)-g(y)| < |f(y)| < \frac{\varepsilon}{2}$, where the first inequality of (2.3) was used. Differentiating (2.3)

$$f'(y) - g'(y) = f'(y)\eta\left(\frac{y-x}{\delta}\right) + f(y)\eta'\left(\frac{y-x}{\delta}\right)\frac{1}{\delta}$$

Applying the second equation in (2.3), the inequality (2.3) and that M is the maximumof |η'| leads to

$$|f'(y) - g'(y)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4M}|y-x|\frac{M}{\delta}$$

.Since $y \in [x-\delta, x+\delta]$, we have$|y-x| \leq \delta$, hence the previous estimate can be continued as

$$|f'(y) - g'(y)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4M}\delta\frac{M}{\delta} = \frac{\varepsilon}{2}$$

. Hence the desired estimates in (2.3) are proved.

Proposition 2.2.3

If all the equilibria of the differential equation $\dot{x} = f(x)$ are hyperbolic, then there are at most finitely many of them in [0,1].

Proof.

Assume that there are infinitely many equilibria in [0,1]. Then one can choose a convergent sequence of equilibria tending to a point $x \in [0,1]$.Then since f is continuously differentiable we have $f(x) = 0$and $f'(x) = 0$, that is x is not a hyperbolic equilibria, which is a contradiction.

Now we turn to the characterisation of one dimensional structurally stable systems.

Theorem 2.2.4.

The dynamical system belonging to the function $f \in X$ is structurally stable, if and only if all equilibria of f are hyperbolic, that is $f \in G$. Moreover, the set G of structurally stable systems is open and dense in the space X.

Proof.

Assume first that f is structurally stable and prove $f \in G$. Since f is equivalent to the systems in a neighbourhood and G is dense, there exists in this neighbourhood a function$g \in G$. Hence all the roots of g are hyperbolic, implying that there are finitely many of them, therefore the equivalence of f and g implies that f has finitely many roots. We show that all of them are hyperbolic. Since the roots are isolated, if one of them were not be hyperbolic, then according to Proposition 2.2.2 an arbitrarily small C^1 perturbation would make it constant zero. That would mean that arbitrarily close to f there is a function, which is zero in an interval, hence it is not equivalent to f contradicting to the fact that f is structurally stable. This proves the first implication.

Assume now that $f \in G$ and prove that f is structurally stable. Proposition 2.2.3 yields that f has finitely many roots. If it has no zeros at all, then the functions close to f in the C^1norm cannot have zeros, hence they are equivalent to f. If f has zeros, then it can be easily seen that functions close to f has the same number of zeros and the sign changes at the zeros are the same as those for f. This implies that their phase portraits are equivalent to that belonging to f.

2.3 Structural stability of higher dimensional systems

In the previous section it was shown that in a one dimensional system the phase portrait can change only in a neighbourhood of non-hyperbolic equilibrium. In two dimensional systems there are bifurcations that are not related to equilibria, namely, the fold bifurcation of periodic orbits, the homoclinic and heteroclinic bifurcation.

Theorem 6.8.(Peixoto).

The dynamical system given by the vector field $f \in X$ isstructurally stable if and only if

- there are finitely many equilibria and all of them are hyperbolic,
- there are finitely many periodic orbits and all of them are hyperbolic,

- there are no trajectories connecting saddle points (heteroclinic or homoclinic orbits).

Moreover, the set of structurally stable systems is open and dense in the space X. We note that the theorem was proved in a more general way for compact two dimensional manifolds. For this general case an extra assumption is needed for structural stability. Even in the case of a torus one can give a structurally unstable system that do not violates any of the assumption of the theorem. In the general case it has to be assumed that the non-wandering points can only be equilibria or periodic points. The general formulation of the theorem can be found in Perko's book [16] and in the book by Wiggins [12].

Based on the cases of one and two dimensional systems one can formulate a conjecture

about the structural stability of systems with arbitrary dimension. This conjecture motivated the following definition of Morse - Smale systems.

Definition 2.3.1

A dynamical systems is called a Morse–Smale system, if

• there are finitely many equilibria and all of them are hyperbolic,

• there are finitely many periodic orbits and all of them are hyperbolic,

• their stable and unstable manifolds intersect each other transversally,

• the non-wandering points can only be equilibria or periodic points.

It can be shown that Morse–Smale system are structurally stable, however, the opposite is not true for more than two dimensional systems, as it was shown by Smale in 1966. If the phase space is at least three dimensional, then there are structurally stable systems with strange attractors as non-wandering sets, containing chaotic orbits. Moreover, it can be proved that in the case of at least three dimensional systems the space of C^1systems contains open sets containing only structurally unstable systems. That is the structurally stable systems do not form a dense and open subset, moreover, the set of them cannot be given as the intersection of open dense sets, that is the structurally stable systems are not generic among three or higher dimensional systems. This means that topological equivalence does not divide the space of C^1systems into open sets.as it is shown in Figure 2.1.

The investigation of structural stability in higher dimensional systems is based on a two dimensional map introduced by Smale and named as Smale horseshoe. This map is dealt with in detail in the book by Guckenheimer and Holmes [3] and in Wiggins's monograph [12].

III. Local Theory

3.1 The Implicit Function Theorem

One of the most important analytic tools for the solution of a nonlinear problem,

$$F(x,y) = 0 \tag{3.1}$$

where$F$ is a mapping $F : U \times V \rightarrow Z$ with open sets $U \subset X, V \subset Y$, andwhere $X, Y, Z$are (real) Banach spaces, is the following **Implicit FunctionTheorem**:

Theorem 3.1.1

Let (3.1) have a solution $(x_0, y_0) \in U \times V$such that the Fr´echet derivative of F with respect to x at (x_0, y_0) is bijective:

$$F(x_0, y_0) = 0,$$

$D_xF(x_0, y_0) : X \rightarrow Z$is bounded (continuous) (3.2)

with a bounded inverse (Banach's Theorem).

Assume also that F and $D_xF$are continuous:

$$F \in C(U \times V, Z),$$

$$D_xF \in C\big(U \times V, L(X,Z)\big), where\ L(X,Z) \tag{3.3}$$

denotes the Banach space of bounded linear operatorsfrom X into Z endowed with the operator norm.

Then there is a neighborhood $U_1 \times V_1 in U \times V\ of\ (x_0, y_0)$and a mapping$f : V_1 \rightarrow U_1 \subset X$such that$f(y_0) = x_0$,

$$F(f(y), y) = 0\ for\ all\ y \in V_1 \tag{3.4}$$

Furthermore, f is continuous on V1:

$$f \in C(V_1, X) \tag{3.5}$$

Finally, every solution of (3.1) in $U_1 \times V_1$is of the form $(f(y), y)$.

Let us consider Y as a space of parameters and X as a space of configuration (a phase space, for example). Then the Implicit Function Theorem allows the following interpretation: The configuration described by problem (3.1) persists for perturbed parameters if it exists for some particular parameter, and it depends smoothly and in a unique way on the parameters. In other words, this theorem describes what one expects: A small change of parameters entails a unique small change of configuration. Thus "dramatic" changes in configurations for specific parameters can happen only if the assumptions of Theorem 3.1 are violated, in particular, if

$D_xF(x_0, y_0): X \rightarrow Z$is not bijective (3.6)

Bifurcation Theory can be briefly described by the investigation of problem(3.1) in a neighborhood of (x_0, y_0) where (3.6) holds.

For later use we need the following addition to Theorem 3.1:

If the mapping F in (3.1) is k-times continuously differentiable on $U \times V$,

i.e.,$F \in C^k(U \times V, Z)$, then the mapping $f$in (3.4) is also k-times continuously differentiable on V_1; i.e., $f \in C^k(V_1, X), k \geq 1$.

If the mapping F is analytic, then the mapping f is also analytic.

For a proof we refer again to [13].

3.2. The Method of Lyapunov–Schmidt

This method describes the reduction of problem (3.1) (which is high- or infinite-dimensional) to a problem having only as many dimensions as the defect (3.6). To be more precise, we need the following definition:

Definition 3.2.1

A continuous mapping $F : U \to Z$, where $U \subset X$is openand where X,Z are Banach spaces, is a nonlinear Fredholm operator if it is Fr´echet differentiable on U and if $DF(x)$fulfills the following:

(i) $dimN(DF(x)) < \infty \ (N = kernel)$,

(ii) $codimR(DF(x)) < \infty \ (R = range)$,

(iii)$R(DF(x))$ $is\ closed\ in\ Z$.

The integer $dimN(DF(x)) - codimR(DF(x))$ is called the Fredholm index of $DF(x)$.

Theorem 3.2.2

There is a neighborhood $U_2 \times V_2$ of $(x_0, y_0))$ in $U \times V \subset X \times Y$such that the problem

$$F(x,y) = 0\ for\ (x,y) \in U_2 \times V_2 \qquad 3.2.1$$

is equivalent to a finite-dimensional problem

$\Phi(v,y) = 0\ for\ (v,y) \in \widetilde{U}_2 \times V_2 \subset N \times Y$, where

$\Phi : \widetilde{U}_2 \times V_2 \to Z_0$is continuous

and

$$\Phi(v_0, y_0) = 0, (v_0, y_0) \in \widetilde{U}_2 \times V_2 \qquad 3.2.2$$

The function Φ, called a bifurcation function.

3.3 An Implicit Function Theorem for One-Dimensional Kernels: Turning Points

In this section we consider mappings $F : U \times V \to Z$with open sets $\subset X, V \subset Y$, where X and Z are Banach spaces, but where this time $Y = R$.

Following a long tradition, we change the notation and denote parameters in R by λ.

We assume$F(x_0,\ \lambda_0) = 0$ for some $(x_0,\ \lambda_0) \in U \times V$,

$$dimN(D_xF(x_0, \lambda_0) = 1 \quad (3.3.1)$$

Obviously, the Implicit Function Theorem, Theorem 3.1.1, is not directly applicable.

We assume now the hypotheses of the Lyapunov−Schmidt reduction for F with the additional assumption that the Fredholm index of $D_xF(x_0, \lambda_0)$ is zero;

i.e., by (3.31)

$$codimR\big(D_xF(x_0, \lambda_0)\big) = 1 \qquad (3.3.2)$$

Since $Y = R$, we can identify the Fr´echet derivative $D_\lambda F(x, \lambda)$ with anelement of Z, namely, by

$$D_\lambda F(x, \lambda)1 = D_\lambda F(x, \lambda) \in Z, 1 \in R \qquad (3.3.3)$$

Theorem 3.3.1

Assume that F : U × V → Z is continuously differentiableon $U \times V \subset X \times R$,

i.e.,$F \in C^1(U \times V, Z)$, (3.3.4) and (3.3.1), (3.3.2), (3.3.3), and that

$$D_\lambda F(x_0, \lambda_0)) \notin R\big(D_xF(x_0, \lambda_0)\big) \qquad (3.3.5)$$

Then there is a continuously differentiable curve through (x_0, λ_0); that is, thereexists

$$\{\big(x(s), \lambda(s)\big) | s \in (-\delta, \delta), \big(x(0), \lambda(0)\big) = (x_0, \lambda_0)\} \qquad (3.3.6)$$

such that

$$F\big(x(s), \lambda(s)\big) = 0\ for\ s \in (-\delta, \delta), \qquad (3.3.7)$$

and all solutions of F(x, λ) = 0 in a neighborhood of (x_0, λ_0) belong to thecurve (3.3.6).

Corollary 3.3.2

The tangent vector of the solution curve (3.3.6) at $(x_0, \lambda_0)$is given by

$$(\hat{v}_0, 0) \in X \times R \qquad (3.3.8)$$

i.e., (3.3.6) is tangent at (x_0, λ_0) to the one-dimensional kernel of $D_xF(x_0, \lambda_0)$.

3.4. Center Manifold Theory

The Hartman-Grobman Theorem, which showed that, in a neighborhood of a hyperbolic critical point $x_0 \in E$, the nonlinear system

$$\dot{X} = f(x) \qquad (3.4.1)$$

is topologically conjugate to the linear system

$$\dot{X} = AX \qquad (3.4.2)$$

with $A = Df(x_0)$, in a neighborhood of the origin. The Hartman-Grobman Theorem therefore completely solves the problem of determining the stability and qualitative behavior in a neighborhood of a hyperbolic critical point of a nonlinear system.

In this section, we present the Local Center Manifold Theorem, which generalizes Theorem to higher dimensions and shows that the qualitative behavior in a neighborhood of a non hyperbolic critical point x_0 of the nonlinear system (3.4.1) with $x \in R^s$ is determined by its behavior on the center manifold near x_0. Since the center manifold is generally of smaller dimension than the system (3.4.1), this simplifies the problem of determining the stability and qualitative behavior of the flow near a non hyperbolic critical point of (3.4.1).

Theorem 3.4.1. (The Local Center Manifold Theorem).

Let $f \in C'(E)$, where E is an open subset of R" containing the origin and $r > 1$. Suppose that $f(0) = 0$ and that $Df(0)$ has c eigenvalues with zero real parts and s eigenvalues with negative real parts, where $c + s = n$. The system (3.4.1) then can be written in diagonal form $\dot{x} = Cx + F(x,y)$

$\dot{y} = Py + G(x,y)$,

where $(x, y) \in R^c \times R^s$, C is a square matrix with c eigenvalues having zero real parts, P is a square matrix with s eigenvalues with negative mat parts, and $F(0) = G(0) = 0$, $DF(O) = DG(O) = 0$; furthermore, there exists $a\,\delta > 0$ and a function $h \in C'(N_\delta(0))$ that defines the local center manifold

$$W_{loc}^c(0) = \{(x, y) \in R^c \times R^s \,/\, y = h(x)\, for\, |x| < \delta\}$$

and satisfies

$$Dh(x)[Cx + F(x, h(x))] - Ph(x) - G(x, h(x)) = 0$$

for $|x| < \delta$; and the flow on the center manifold W°(0) is defined by the system of differential equations

$$\dot{x} = Cx + F(x, h(x))$$

for all $x \in R^s$ with $|x| < \delta$.

3.5. Normal Form Theory

The Hartman-Grobman Theorem shows us that in a neighborhood of a hyperbolic critical point, the qualitative behavior of a nonlinear system

$$\dot{X} = f(x) \qquad (3.5.1)$$

where $x \in R^n$ is determined by its linear part. we know that the linear part of (3.5.1) can be put into Jordan canonical form $\dot{x} = Jx$, which makes it easy to solve the linear system. The Local Center Manifold Theorem in the previous section showed us that, in a neighborhood of a nonhyperbolic critical point, determining the qualitative behavior of (3.5.1) could be reduced to the problem of determining the qualitative behavior of the nonlinear system

$$\dot{x} = Jx + F(x) \qquad (3.5.2)$$

on the center manifold. Since the dimension of the center manifold is typically less than n, this simplifies the problem of determining the qualitative behavior of the system (3.5.1) near a non hyperbolic critical point. However, analyzing this system still may be a difficult task. The normal form theory allows us to simplify the nonlinear part, F(x), of (3.5.2) in order to make this task as easy as possible. This is accomplished by making a nonlinear, analytic transformation of coordinates of the form

$$x = y + h(y), \qquad (3.5.3)$$

where $h(y) = 0(|y|2)$ as $|y| \to 0$.

IV. Global Theory

4.1 A Global Implicit Function Theorem

We consider a continuous mapping $F : X \times R \to Z$, where $X \subset Z$is continuouslyembedded, and we assume a solution

$F(x_0, \lambda_0) = 0.$ (4.1.1)

Apart from the assumptions (3.1.2) and (3.1.3) for the local Implicit FunctionTheorem, Theorem 3.1.1, we need a setting such that a degree for $F(\cdot, \lambda)$can be defined.

The Leray–Schauder degree is applicable if $X = Z, F(x, \lambda) = x + f(x, \lambda)$,and $f : X \times R \to X$is completely continuous. The degree for Fredholm operators can be used if $F \in C^2(X \times R, Z), D_x F(x, \lambda) \in L(X, Z)$ is admissibleaccording to the following Definition 4.1.1, and F is proper on every closed and boundedsubset of X × R according to the following Definition 4.1.2(iii).

Definition 4.1.1

A class of linear operators $A \in L(X, Z)$ is called admissible if the following hold:

(i) A is a Fredholm operator of index zero.

(ii) $A : Z \to Z$with domain of definition $D(A) = X$is closed.

(iii) The spectrum $\sigma(A)$ in a strip $(-\infty, c) \times (-i\varepsilon, i\varepsilon) \subset C$ for some $c > 0$, ε >0 consists of finitely many eigenvalues of finite algebraic multiplicity.

Their total number (counting multiplicities) in that strip is stable under small perturbations in the class of A in $L(X, Z)$.

Definition 4.1.2

Let $U \subset X$be open and bounded.

An operator $F : U \to Z$is called admissible if

(i) $F \in C^2(\widehat{U}, Z), U \subset \widehat{U}$,where$\widehat{U}$is open in X,

(ii) The class $\{DF(x) | x \in \widehat{U}\} \subset L(X, Z)$is admissible according to DefinitionII.5.1,

(iii) F is proper; i.e., the inverse image in U of a compact set in Z is compact in X.

A **Global Implicit Function Theorem** then reads as follows:

Theorem 4.1.3

Assume the preceding properties of $F : X \times R \to Z$and (4.1) such that

$$D_x F(x_0, \lambda_0) \in L(X, Z)$$

is bijective and

$D_x F \in C(B_r(x_0) \times (\lambda_0 - \rho, \lambda_0 + \rho), L(X, Z))$ for some $r, \rho > 0$ (4.1.2)

Let S denote the set of all solutions of $F(x, \lambda) = 0\ in\ X \times R$and let Cbe the (connected) component of S that contains the local solution curve

$\{(x(\lambda), \lambda) | \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)\}$ through $(x_0, \lambda_0) = (x(\lambda_0), \lambda_0)$ given by Theorem3.1.1. Then

(i) $C = \{(x_0, \lambda_0)\} \cup C^+ \cup C^-, C^+ \cap C^- = \emptyset$, and $C^+, C^-$are each unbounded,or

(ii) $C \backslash \{(x_0, \lambda_0)\}$ is connected

Proof.

Assume that $C \backslash \{(x_0, \lambda_0)\}$is not connected. Then$C = \{(x_0, \lambda_0)\} \cup C^+ \cup C^-$, where $C^+ \cap C^- = \emptyset$, and let C^+ denote the component of

$$\{(x(\lambda), \lambda) | \lambda \in (\lambda_0, \lambda_0 + \delta)\}\ in\ S \backslash \{(x_0, \lambda_0)\}.$$

Assume that C^+ is bounded.

By known theorem, the bounded and closed set$C^+ \cup \{(x_0, \lambda_0)\}. = C^+$ iscompact.

we can construct a bounded open set $U \subset X$ × R such that

$$C^{+} \subset U \text{ and } \partial U \cap S = \{(x_0, \lambda_0)\} \quad (4.1.3)$$

Setting as before $U_\lambda = \{x \in X | (x, \lambda) \in U\}$, we can also assume that
$\bar{U}_{\lambda_0} \cap \bar{B}_r(x_0) = \emptyset$ and $\bar{U}_{\lambda_0} \cup \bar{B}_r(x_0) = (\bar{U})_{\lambda_0}$ (4.1.4) where $r > 0$ is so small that $x = x_0$ is an isolated solution of $F(x, \lambda_0) = 0$
in $\bar{B}_r(x_0)$. (Note that $(4.1.4)_1$ denotes the closure of the fiber in X, whereas (4.1.4)2 denotes the fiber of the closure in $X \times R$.)Then the additivity and the homotopy invariance of the respective degree imply

$$d(F(\cdot,\lambda_0), B_r(x_0), 0) + d\big(F(\cdot,\lambda 0), U_{\lambda_0}, 0\big) = d(F(\cdot,\lambda), U_\lambda, 0) for\ \lambda \geq \lambda_0,$$
$$= 0, since\ U_\lambda = \emptyset for\ large\ \lambda > \lambda_0 \quad (4.1.5).$$

On the other hand, if $U\lambda_0 \neq \emptyset$,

$$d(F(\cdot,\lambda_0), U_{\lambda_0}, 0) = d(F(\cdot,\lambda), U_\lambda, 0)\ for\ \lambda \leq \lambda_0,$$
$$= 0, \text{ since } U_\lambda = \emptyset \text{ for large } \lambda < \lambda_0 \quad (4.1.6)$$

This proves $(d(F(\cdot,\lambda_0), B_r(x_0), 0) = 0$ (4.1.7)
But $D_x F(x_0, \lambda_0) \in L(X, Z)$ is bijective and the local degree (4.1.7) is the
index $i(F(\cdot,\lambda_0), x_0) \in \{-1, 1\}$; This contradictionproves that C^+ is unbounded, and the unboundedness of $C^-$is provedin the same way.

Remark 4.1.4

The proof of Theorem 4.1.3 shows that the assumption (4.1.2) can be reduced to $(4.1.2)_1$. Note that the local Implicit Function Theorem does not hold without assumption $(II.6.2)_2$, which means that there is not necessarily a unique local curve of solutions {(x(λ), λ)} through (x_0, λ_0).

But the nonzero local degree (4.1.7) and the homotopy invariance of the degree imply that the solution (x_0, λ_0) is continued for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Let C denote the component in S containing (x_0, λ_0). Then the same alternative (i), (ii) holds, and in any case, $C \backslash \{(x_0, \lambda_0)\} = \emptyset$.

Remark 4.1.5

The possibility of a global extension of the local solution curve given by the Implicit Function Theorem is also called **Global Continuation**. It gives solutions of $F(x, \lambda) = 0$for all λ ∈R, provided that (x_0, λ_0) is the only solution for $\lambda = \lambda_0$and that there is an a priori estimatefor solutions x for all λ in finite intervals of R. This possibility motivated Leray and Schauder to extend the Brouwer degree to infinite dimensions in order to solve nonlinear elliptic partial differential equations.

V. Applications

5.1. Center Manifold Reduction

A center manifold at a given non hyperbolic equilibrium is an invariant manifold of the considered differential equation which is tangent at the equilibrium point to the (generalized) eigen space of the neutrally stable eigenvalues. As the local dynamic behavior *transverse* to the center manifold is relatively simple, the potentially complicated asymptotic behaviours of the full system are captured by the flows restricted to the center manifolds.

Center manifold theory plays an important role in the study of the stability of dynamical systems when the equilibrium point is not hyperbolic. The combination of this theory with the normal form approach was used extensively to study parameterized dynamical systems exhibiting bifurcations. The center manifold theorem provides, in this case, a means of systematically reducing the dimension of the state spaces which need to be considered when analyzing bifurcations of a given type. In fact, after determining the center manifold, the analysis of these parameterized dynamical systems is based only on the restriction of the original system on the center manifold whose stability properties are the same as the ones of the full order system.

5.2. Normal form theory

Normal forms theory provides one of the most powerful tools in the study of nonlinear dynamical systems, in particular, in the stability and bifurcation analysis. In the context of finite-dimensional ordinary differential equations (ODEs), this theory can be traced back as far as Euler. The basic idea of normal form consists of employing successive, near-identity ,nonlinear transformations, which leads to a differential equation in a simpler form, qualitatively equivalent to the original system in the vicinity of a fixed equilibrium point, thus hopefully greatly simplifying the dynamics analysis.

As we develop the method, three important characteristics should become apparent. (i) The method is local in the sense that the coordinate are generated in a neighborhood of a known solution. For our purposes, the known solution will be an equilibrium. (ii) In general, the coordinate transformations will be nonlinear of the dependent variables. However, the important point is that coordinate transformations are found by solving a sequence of problems. (iii) The structure of the norm form is determined entirely by the linear part of the vector field. A key notion in normal form reduction is that of resonance. In particular, the Jacobian matrix of the system, evaluated at the equilibrium point determines which monomials in the formal expansion of the system are resonant and cannot be removed by any smooth coordinate transformation.

5.3. Lyapunov-Schmidt Reduction

Generally, particular types of solutions of a differential equation, such as a fixed point, relative equilibrium, or a periodic orbit can be found by determining the zeros of an appropriate map F and applying the Lyapunov-Schmidt procedure. The Lyapunov-Schmidt reduction results in the so-called bifurcation equations, a finite set of equations, equivalent to the original problem. This finite set of equations may inherit the symmetry properties of the original system if the reduction is done properly.

For example, if we are looking for periodic solution, the map F has a natural symmetry group S^1 representing phase shifts along the periodic solution.It would be interesting to know for what values of parameter, say, solutions of the bifurcation equation disappear or are created. These particular values of are called bifurcation values.

It provides a classification of the various cases based on co dimension. The reason this is possible is that the co dimension k sub manifolds in the space of all smooth functions having zeros can be described algebraically by imposing conditions on derivatives of the functions. This gives us a way of classifying the various possible bifurcations and of computing the proper un foldings. Lyapunov-Schmidt reduction is a very effective method to investigate the phenomenon of Hopf bifurcation, which concerns the birth of a periodic solution from an equilibrium solution through a local oscillatory instability.

5.4. Degree theory

Many applications, including some bifurcation problems of functional differential

equations, lead to the problem of finding all zeros of a mapping $f: U \subseteq X \rightarrow X,$ where X is some (real) Banach space. The basic idea of degree theory is as follow. Given a (sufficiently smooth) domain U with enclosing Jordan curve ∂U, we have defined a degree

$$deg(f; U;\ z_0) = n(f(\partial U);\ z_0) = n(f(\partial U) - z_0;\ 0) \in Z,$$

which counts the number of solutions of $f(z) = z_0$ inside U. The invariance of this degree with respect to certain deformations of $f$allowed us to explicitly computed eg $(f; U;\ z_0)$ even in nontrivial cases. Degree theory has been developed for various classes of mappings. Generalized Brouwer degree theory to an infinite Banach space and established the so-called the Leray-Schauder degree. It turns out that the Leray-Schauder degree is very powerful tool in proving various existence results for nonlinear differential equations.

There are several important application areas of bifurcation theory and this theory has also been applied in the study of several theoretical examples which are difficult to access experimentally.

In recent years many types of bifurcations of flow and maps have been studied and classified including saddle node, Hopf, umbilic, zip, homoclinic tangencies, period doubling and

cusp bifurcations. It is our belief that in the years to come the bifurcation theory plays a more active role in various application domains of Science and Technology.

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