A COMMON FIXED POINT THEOREM IN 2-BANACH SPACE FOR NON EXPANSIVE MAPPING

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Abstract

In this paper our aim is to discuss about fixed point theory in 2-Banach Space through the concept of non expansive mapping.

Keywords: Normed space , 2-Normed Space, 2- Banach Space, Non expansive mapping.

1. INTRODUCTION

Let X be a set and F be a function from X to X. A fixed point of F is an element $x \in X$, such that F(x) =x. A fixed point theorem is a theorem that assets that every function that satisfies some given property must be a fixed point.

The concept of two Banach Space firstly introduced by Gahler[2] .This space was subsequently been studied by Mathematician (Kirk 1981). Recently A.S.Saluja, A.K.Dhakde [5], Shefali vijay vargiya,sonal Bharti [6] and Shukla D.P, Vivek Tiwari [7] also worked Common fixed point theorem in 2-Banach Spaces for self mapping and D.K.Mali, R.K.Gujetiya, Mala Hakwadhiya [1] and V.Gupta,A.K.Tripathi [8] also proved fixed point theorem in 2- Banach Space for non expansive mapping. In the paper I prove a common fixed point theorem in 2 – Banach Space by taking non expansive mapping.

2. PRELIMINARIES

Definition 2.1 [1]

Let X be a real linear space and $\|.,.\|$ be a non negative real valued function defined on X satisfying the following conditions.

(i) $\|x, y\| = 0$ if and only if x and y are linear independent.

(ii) $\|x, y\| = \|y, x\|$ for all $x, y \in X$

(iii) $\|x, ay\| = |a|\|x, y\|$, a being real for all $x, y \in X$

(iv) $\|x, y+z\| \le \|x, y\| + \|x, z\|$, for all $x, y, z \in X$

Then $\|\cdot,\cdot\|$ is called a 2-norm and the pair (X, $\|\cdot,\cdot\|$) is called a linear 2-normed space. So a 2-norm $\|x, y\|$ always satisfies $\|x, y+ax\| = \|x, y\|$ for all $x, y \in X$ and all scalar a.

Definition 2.2 [1]

A sequence $\{x_n\}$ in a 2- normed linear space (X, $\|.,.\|$) is said to be convergent sequence if there is a point $x \in X$ such that $\|x_n - x, a\| = 0$ as $n \to \infty$ and for all $a \in X$ .If x_n converges to x ,we write $x_n \to x \; as \; n \to \infty$.

Definition 2.3 [1]

A sequence $\{x_n\}$ in a 2-normed space (X, $\|.,.\|$) is called a Cauchy sequence if $\|x_m - x_n, a\| = 0$ as $m, n \to \infty$ and for all $a \in X$.

Definition 2.4 [1]

A linear 2 normed space is said to be complete if every Cauchy sequence is convergent to an element on X.A complete 2-normed space X is called 2- Banach Spaces.

Definition 2.5 [1]

Let X be a 2-Banach space and T be a Self mapping of X .T is said to be continuous at x if every sequence $\{x_n\}$ in X, $\{x_n\} \to x \, as \, n \to \infty$ implies $\{T(x_n)\} \to T(x) \; as \; n \to \infty$.

Definition 2.6 [1]

Let X be a 2-Banach space and C be a non empty bounded closed and convex subset of X. A mapping $T : C \to X$ is said to be non expansive if $\|T(x) - T(y), a\| \le \|x - y, a\|$ where $x, y \in C$.

3. THEOREM

Let F and G be two non expansive mapping of a 2-Banach space X into itself. F and G satisfy the following condition:

(1). FG=GF=I, where I is identity mapping.

(2). $$\|Fx - Gy, a\| \le \alpha \left[\frac{\|x - Gy, a\| \|y - Fx, a\|}{\|x - Fx, a\| + \|x - y, a\|} \right] + \beta \left[\frac{\|x - Fx, a\| \|y - Fy, a\| \|x - Gy, a\| + \|x - y, a\|^3}{\left[\|y - Gy, a\| + \|x - y, a\| \right]^2} \right]$$

$$+ \gamma \left[\|x - Fx, a\| + \|y - Gy, a\| \right] + \delta \|x - Gy, a\| + \|y - Fx, a\| + \eta \|x - y, a\|$$

Where $\alpha, \beta, \gamma, \delta, \eta \ge 0$ for all $x, y \in X$, Where $\alpha + 5\beta + 4\gamma + 4\delta + \eta \le 2$ and $\alpha + \beta + 2\delta + \eta < 1$ .Then F and G have a unique common fixed point.

Proof:

Suppose x is any point in 2 Banach space X.

Taking y= $\frac{1}{2}(F + I)x$, z=G(y) and u=2y-z then

$$\|z - x, a\| = \|Gy - GFx, a\|$$

Now using (1) and (2) ,we have

$$\|z-x,a\| \le \alpha\left[\frac{\|y-GFx,a\|\|Fx-Gy,a\|}{\|y-Gy,a\|+\|y-Fx,a\|}\right]+\beta\left[\frac{\|y-Gy,a\|\|Fx-GFx,a\|\|y-GFx,a\|+\|y-Fx,a\|^3}{[\|Fx-GFx,a\|+\|y-Fx,a\|]^2}\right]$$

$$+\gamma[\|y-Gy,a\|+\|Fx-GFx,a\|]+\delta[\|y-GFx,a\|+\|Fx-Gy,a\|]+\eta\|y-Fx,a\|$$

$$\le \alpha\left[\frac{\|y-x,a\|\|Fx-Gy,a\|}{\|Fx-Gy,a\|}\right]+\beta\left[\frac{\|y-Gy,a\|\|Fx-x,a\|\|y-x,a\|+\|y-Fx,a\|^3}{[\|Fx-x,a\|+\|y-Fx,a\|]^2}\right]$$

$$+\gamma[\|y-Gy,a\|+\|Fx-x,a\|]+\delta[\|y-x,a\|+\|Fx-y,a\|+\|y-Gy,a\|]+\frac{\eta}{2}\|x-Fx,a\|$$

$$\le \frac{\alpha}{2}\|x-Fx,a\|+\beta\left[\frac{\|y-Gy,a\|\|Fx-x,a\|\frac{1}{2}\|x-Fx,a\|+\frac{1`}{8}\|x-Fx,a\|^3}{\|x-y,a\|^2}\right]$$

$$+\gamma[\|y-Gy,a\|+\|Fx-x,a\|]+\delta\left[\frac{1}{2}\|x-Fx,a\|+\frac{1}{2}\|x-Fx,a\|+\|y-Gy,a\|\right]+\frac{\eta}{2}\|x-Fx,a\|$$

$$\le \frac{\alpha}{2}\|x-Fx,a\|+\beta\left[\frac{\frac{1}{2}\|x-Fx,a\|^2\|y-Gy,a\|+\frac{1`}{8}\|x-Fx,a\|^3}{\frac{1}{4}\|x-Fx,a\|^2}\right]$$

$$+\gamma[\|y-Gy,a\|+\|Fx-x,a\|]+\delta[\|x-Fx,a\|+\|y-Gy,a\|]+\frac{\eta}{2}\|x-Fx,a\|$$

$$\le \frac{\alpha}{2}\|x-Fx,a\|+\beta\left[2\|y-Gy,a\|+\frac{1`}{2}\|x-Fx,a\|\right]$$

$$+\gamma[\|y-Gy,a\|+\|Fx-x,a\|]+\delta[\|x-Fx,a\|+\|y-Gy,a\|]+\frac{\eta}{2}\|x-Fx,a\|$$

$$\le\left(\frac{\alpha}{2}+\frac{\beta}{2}+\gamma+\delta+\frac{\eta}{2}\right)\|x-Fx,a\|+(2\beta+\gamma+\delta)\|y-Gy,a\|\_\_\_\_\_\_\_\_(3)$$

$$\|u-x,a\|=\|2y-z-x,a\|=\|Gy-Fx,a\|$$

Now using (1) and (2)

$$\|Gy-Fx,a\| \le \alpha\left[\frac{\|y-Fx,a\|\|x-Gy,a\|}{\|y-Gy,a\|+\|y-x,a\|}\right]+\beta\left[\frac{\|y-Gy,a\|\|x-Fx,a\|\|y-Fx,a\|+\|x-y,a\|^3}{[\|x-Fx,a\|+\|y-x,a\|]^2}\right]$$

$$+\gamma[\|x-Fx,a\|+\|y-Gy,a\|]+\delta[\|y-Fx,a\|+\|x-Gy,a\|]+\eta\|y-x,a\|$$

$$\le \alpha\left[\frac{\|y-Fx,a\|\|x-Gy,a\|}{\|x-Gy,a\|}\right]+\beta\left[\frac{\|y-Gy,a\|\|x-Fx,a\|\frac{1}{2}\|x-Fx,a\|+\frac{1}{8}\|x-Fx,a\|^3}{[\|y-Fx,a\|]^2}\right]$$

$$+\gamma\left[\|x-Fx,a\|+\|y-Gy,a\|\right]+\delta\left[\frac{1}{2}\|x-Fx,a\|+\|x-y,a\|+\|y-Gy,a\|\right]+\frac{\eta}{2}\|x-Fx,a\|$$

$$\leq\frac{\alpha}{2}\|x-Fx,a\|+\beta\left[\frac{\frac{1}{2}\|y-Gy,a\|\|x-Fx,a\|^2+\frac{1}{8}\|x-Fx,a\|^3}{\frac{1}{4}\|x-Fx,a\|^2}\right]$$

$$+\gamma\left[\|x-Fx,a\|+\|y-Gy,a\|\right]+\delta\left[\frac{1}{2}\|x-Fx,a\|+\frac{1}{2}\|x-Fx,a\|+\|y-Gy,a\|\right]+\frac{\eta}{2}\|x-Fx,a\|$$

$$\leq\frac{\alpha}{2}\|x-Fx,a\|+\beta\left[2\|y-Gy,a\|+\frac{1}{2}\|x-Fx,a\|\right]$$

$$+\gamma\left[\|x-Fx,a\|+\|y-Gy,a\|\right]+\delta\left[\|x-Fx,a\|+\|y-Gy,a\|\right]+\frac{\eta}{2}\|x-Fx,a\|$$

$$\leq\left(\frac{\alpha}{2}+\frac{\beta}{2}+\gamma+\delta+\frac{\eta}{2}\right)\|x-Fx,a\|+(2\beta+\gamma+\delta)\|y-Gy,a\| \qquad (4)$$

$$\|z-u,a\|\leq\|z-x,a\|+\|x-u,a\|$$

Therefore by using 3 and 4, we get

$$\|z-u,a\|\leq(\alpha+\beta+2\gamma+2\delta+\eta)\|x-Fx,a\|+(4\beta+2\gamma+2\delta)\|y-Gy,a\|$$

$$\|z-u,a\|=\|Gy-2y+z,a\|=2\|y-Gy,a\|$$

$$2\|y-Gy,a\|\leq(\alpha+\beta+2\gamma+2\delta+\eta)\|x-Fx,a\|+(4\beta+2\gamma+2\delta)\|y-Gy,a\|$$

$$(2-4\beta-2\gamma-2\delta)\|y-Gy,a\|\leq(\alpha+\beta+2\gamma+2\delta+\eta)\|x-Fx,a\|$$

$$\|y-Gy,a\|\leq\frac{\alpha+\beta+2\gamma+2\delta+\eta}{2-4\beta-2\gamma-2\delta}\|x-Fx,a\|$$

$$\|y-Gy,a\|\leq q\|x-Fx,a\|$$

Where $q=\frac{\alpha+\beta+2\gamma+2\delta+\eta}{2-4\beta-2\gamma-2\delta}\leq 1$ and $\alpha+5\beta+4\gamma+4\delta+\eta\leq 2$

Let $T=\frac{1}{2}(F+I)$ then for any $x\in X$

$$\|T^2(x)-Tx,a\|=\|TT(x)-Tx,a\|$$

$$\|Ty-y,a\|=\left\|\frac{1}{2}(F+I)y-y,a\right\|=\frac{1}{2}\|y-Fy,a\|$$

$\Rightarrow\frac{1}{2}\|FGy-Fy,a\|\leq\frac{1}{2}\|Gy-y,a\|$, because F is nonexpansive function.

$\|T^2(x)-Tx,a\| \le \frac{q}{2}\|x-Fx,a\|$, by definition of q,

we claim that $T^n x$ is a Cauchy sequence in X.

Also by completeness $T^n x$ converges to T(x).

(ie) $\lim_{n\to\infty} T^n x = x_0$

$F(x_0)=x_0$, x_0 is fixed point of F.

Again $\|T^2(x)-T(x),a\| \le \frac{q}{2}\|x-F(x),a\|$

$$\le \frac{q}{2}\|FGx-F(x),a\|$$

$$\le \frac{q}{2}\|Gx-x,a\|$$

We conclude that $G(x_0)=x_0$, x_0 is fixed point of G.

$F(x_0)=G(x_0)=x_0$ so x_0 is a common fixed point of F and G

Uniqueness: Let $y_0(x_0 \ne y_0)$ be an another common fixed point of F and G.

Therefore by using (2),we get

$$\|x_0-y_0,a\| = \|FGx_0-FGy_0,a\| = \|FGx_0-GFy_0,a\|$$

$$\|x_0-y_0,a\| \le \alpha\left[\frac{\|Gx_0-GFy_0,a\|\|Fy_0-FGx_0,a\|}{\|Gx_0-FGx_0,a\|+\|Gx_0-Fy_0,a\|}\right]$$

$$+\beta\left[\frac{\|Gx_0-FGx_0,a\|\|Fy_0-GFy_0,a\|\|Gx_0-GFy_0,a\|+\|Gx_0-Fy_0,a\|^3}{[\|Fy_0-GFy_0,a\|+\|Gx_0-Fy_0,a\|]^2}\right]$$

$$+\gamma[\|Gx_0-FGx_0,a\|+\|Fy_0-GFy_0,a\|]+\delta\|Gx_0-GFy_0,a\|+\eta\|Gx_0-Fy_0,a\|$$

$$\|x_0-y_0,a\| \le \alpha\left[\frac{\|x_0-y_0,a\|\|y_0-x_0,a\|}{\|x_0-x_0,a\|+\|x_0-y_0,a\|}\right]+\beta\left[\frac{\|x_0-x_0,a\|\|y_0-y_0,a\|\|x_0-y_0,a\|+\|x_0-y_0,a\|^3}{[\|x_0-y_0,a\|]^2}\right]$$

$$+\gamma[\|x_0-x_0,a\|+\|y_0-y_0,a\|]+\delta\|x_0-y_0,a\|+\eta\|x_0-y,a\|$$

$$\|x_0-y_0,a\| \le (\alpha+\beta+2\delta+\eta)\|x_0-y_0,a\|$$

Since $(\alpha+\beta+2\delta+\eta)<1$

Which is contradiction, so $x_0=y_0$ Hence x_0 is a unique common fixed point of F and G.

4.CONCLUSION

In this paper, I have discussed common fixed point theorem in 2-Banach Space through concept of non-expansive mapping.

REFERENCES

[1] Dheeraj Kumari Mali, Gujetiya.R.K and Mala Hakwadiya. 2014. A Common fixed point theorem in 2-Banach Space for non expansive mapping. American International Journal of Research in Science,Technology,Engg and Mathematics, 7(3).192-195 .

[2] Gahlar.S .1964. 2-Metreche raume and ihre topologiscche structure, Math Nadh ,.26, 115-148.

[3] Kirk.W.A.1983.Fixed point theorem for non expansive mapping,Contemporary mathematics,18. 121-140.

[4] Kirk.W.A. 1981.Fixed point theorem for non expansive mapping, Lecture notes in mathematics, Springer-verlag,Berlin and newyork, 886,111-120.

[5] Saluja.A.S, Alkesh Kumar Dhakde. 2013. Some fixed point and common fixed point theorem in 2-Banach Space. American Journal of Engineering Research, 2(5). 122-127.

[6] Shefali Vijayvargiya, Sonal Bharti. 2017. Some fixed point result in Banach Spaces. Global Journal of pure and Applied mathematics.13(9) .5871-5890.

[7] Shukla.D.P and Vivek Tiwari.2014. Some fixed point and common fixed point theorem in 2-Banach Spaces. IOSR Journal of Mathematics,10(6). 32-37.

[8] Vishal Gupta, Tripathi.A.K .2011. A fixed point theorem in 2-Banach Space for non expansive mapping. Innovative system Design and Eng.2(4).71-74.