πgs*g-CLOSED SETS AND QUASI s*g-NORMAL SPACES

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Abstract: In this paper, we introduce a new class of sets called πgs*g-closed sets in topological spaces. Also we study and investigate the relationship with other existing closed sets. Moreover, we introduce some functions such as s*g-closed, πgs*g-closed, almost s*g-closed, almost πgs*g-closed, πgs*g-continuous and almost πgs*g-continuous. We also study a new class of normal space called, quasi s*g-normal space. The relationships among normal, π-normal, quasi normal, softly normal, mildly normal, α-normal, πα-normal, quasi α-normal, softly α-normal, mildly α-normal, s*g-normal, πs*g-normal, quasi s*g-normal, softly s*g- normal and mildly s*g-normal spaces are investigated. Further we show that this property is a topological property and it is a hereditary property only with respect to closed domain subspaces. Utilizing πgs*g-closed sets and some functions, we obtained some characterizations and preservation theorems for quasi s*g-normal spaces.

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Keywords : π-open, s*g-open, πgs*g-open, π-closed, s*g-closed, and πgs*g-closed sets; πgs*g-closed, almost πgs*g-closed, πgs*g-continuous and almost πgs*g-continuous functions; quasi s*g-normal spaces.

1. Introduction

In this paper, we introduced the concept of quasi s*g-normal spaces in topological spaces by using s*g- open sets due to M. Khan, T. Noiri and M. Hussain [6] and obtained several properties of such a space. M. Khan [6] introduced the concepts of s*g-closed sets in topological spaces. We introduced the concepts of g s*g-closed, πgs*g-closed sets, s*g-closed, gs*g-closed, πgs*g-closed, almost s*g-closed, almost gs*g-closed, almost πgs*g-closed, πgs*g-continuous and almost πgs*g-continuous functions. Moreover, we obtain some new characterizations and preservation theorems of quasi s*g-normal spaces. In 1968, Zaitsev [14] introduced the concept of quasi-normal space in topological spaces and obtained several properties of such a space. Recently, Hamant Kumar and M.C.Sharma [5] introduced the concept of πgγ-closed sets as weak form of πg-closed sets due to Dontchev [4]. and introduced the concept of quasi γ-normal spaces and by using πgγ-closed sets and obtained a characterization and preservation theorems for quasi γ-normal spaces. Further show that this property is a topological property and it is a hereditary property only with respect to closed domain subspaces. The relationship among of normal , π-normal, p-normal, πp-normal , γ-normal, πγ – normal, quasi **-**normal , quasi p- normal, quasi γ **-**normal , mildly p- normal, mildly γ-normal are investigated .

2. Preliminaries

2.1 Definition. A subset A of a topological space X is said to be

1. **g-closed** [7] if cl(A) ⊂ U whenever A ⊂ U and U is open in X.

2.**α-closed [10]** if cl(int (cl(A))) ⊂ A.

3. **α-open [11]** if A ⊂ int(cl(int(A))).

3. **αg-closed [9]** if α-cl(A) ⊂ U whenever A ⊂ U and U is open in X.

4. **gα-closed [9]** if α-cl(A) ⊂ U whenever A ⊂ U and U is α-open in X.

5. **s-open [8]** if A ⊂ cl(int(A)).

6. **generalized semi-closed [2]** (briefly, gs-closed) if s-cl(A) ⊂ U whenever A ⊂ U and U is open in X.

7. **semi-generalized closed [2]** (briefly, sg-closed) s-cl(A) ⊂ U wnenever A ⊂ U and U is s-open in X.

6. **s*g-closed [6]** if cl(A) ⊂ U whenever A ⊂ U and U is semi-open in X.

7. **gs*g-closed** if s*g-cl(A) ⊂ U whenever A ⊂ U and U is s*g- open in X.

8. **πg -closed** [4] if cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.

9**. πgα-closed [1]** if α-cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.

10. **πgs-closed [3]** if s-cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.

11. **πgs*g-closed** if s*g-cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.

12. The finite union of all regularly open sets is said to be **π - open [14].** The complement of **g-closed** (resp. α-closed ,s*g-closed, gs*g-closed, πg-closed ,πgs*g-closed) set is called **g-open** (resp. α-open, s*g-open, gs*g-open, πg-open, πgs*g-open) set and the complement of π-open is called π-closed. The intersection of all s*g-closed sets containing A is called the **s*g-closure of A** and denoted **s*g-cl**(**A**).The union of all s*g-open subsets of X which are contained in A is called the **s*g-interior of A** and denoted by **s*g-int**(**A**).

We have the following implications for the properties of subsets :

closed	$\Rightarrow$	g-closed	$\Rightarrow$	πg-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
s*g-closed	$\Rightarrow$	gs*g-closed	$\Rightarrow$	πgs*g-closed

Where none of the implications is reversible as can be seen from the following examples.

2.2.Example. Let X = { a, b, c, d } and τ = { ϕ, {a}, {b}, {a, b }, {a, b,c}, X }.Let A ={c}.Then A is πgs*g-closed set but not πg-closed set in X.

2.3. Example. Let X = { a, b, c, d } and τ = { ϕ, {a},{d}, {a, d}, {c, d}, {a, c, d }, X }. Then the set A = {a} is πgs*g-closed set not gs*g-closed set in X .

2.4. Theorem. A subset A of a topological space X is πgs*g-open iff F ⊂ s*g-int (A) whenever F is π-closed and F ⊂ A.

3. Quasi s*g- Normal Spaces

3.1.Definition. A topological space X is said to be **quasi s*g-normal** if for every pair of disjoint π-closed subsets H, K, there exist disjoint s*g-open sets U, V of X such that H ⊂ U and K ⊂ V.

3.2.Example. Let X = { a, b, c, d } and τ = { ϕ, {a}, {c}, {a, c},{ b, d}, {a, b, d}, {b, c, d}, X }. The pair of disjoint π-closed subsets of X are A = {a} and B = {c}. Also U = {a} and V = {b, c, d} are disjoint s*g-open sets such that A ⊂ U and B ⊂ V. Hence X is quasi-normal as well as quasi s*g-normal because every open set is s*g-open set .

By the definitions and examples stated above, we have the following diagram: normal $\Rightarrow$ quasi-normal $\Rightarrow$ quasi s*g-normal.

3.3.Theorem. For a topological space X, the following are equivalent :

(a) X is quasi s*g-normal.

(b) For any disjoint π-closed sets H and K, there exist disjoint gs*g-open sets U and V such that $H \subset U$ and $K \subset V$.

(c) For any disjoint π-closed sets H and K, there exist disjoint $\pi g\alpha$*-open sets U and V such that $H \subset U$ and $K \subset V$.

(a) For any π-closed set H and any π-open set V containing H, there exists a gs*g-open set U of X such that $H \subset U \subset$ s*g-cl(U) $\subset V$.

(b) For any π - closed set H and any π - open set V containing H, there exists a πgs*g- open set U of X such that $H \subset U \subset$ s*g-cl(U) $\subset V$.

Proof. (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (d) $\Rightarrow$ (e) , (c) $\Rightarrow$ (d), and (e) $\Rightarrow$ (a). (a) $\Rightarrow$ (b). Let X be quasi s*g-normal. Let H, K be disjoint π - closed sets of X. By assumption, there exist disjoint s*g-open sets U, V such that $H \subset U$ and $K \subset V$. Since every s*g-open set is gs*g-open, U ,V are gs*g-open sets such that $H \subset U$ and $K \subset V$.

(b) $\Rightarrow$ (c). Let H, K be two disjoint π -closed sets. By assumption, there exist gs*g-open sets U and V such that $H \subset U$ and $K \subset V$. Since gs*g-open set is πgs*g-open, U and V are πgs*g-open sets such that $H \subset U$ and $K \subset V$.

(d) $\Rightarrow$ (e). Let H be any π-closed set and V be any π-open set containing H. By assumption, there exists a gs*g-open set U of X such that

$H \subset U \subset$ s*g-cl(U) $\subset V$. Since every gs*g-open set is πgs*g-open, there exists a πgs*g-open set U of X such that $H \subset U \subset$ s*g-cl(U) $\subset V$.

(c) $\Rightarrow$ (d). Let H be any π-closed set and V be any π-open set containing H. By assumption, there exist πgs*g-open sets U and W such that $H \subset U$ and

$X - V \subset W$. By **Theorem 2.4**, we get $X - V \subset$ s*g-int(W) and

s*g-cl(U) $\cap$ s*g- int(W) = ϕ.Hence $H \subset U \subset$ s*g-cl(U) $\subset X -$ s*g-int(W) $\subset V$.

(e) $\Rightarrow$ (a). Let H, K be any two disjoint π -closed set of X. Then $H \subset X - K$ and $X - K$ is π-open. By assumption, there exists a πgs*g-open set G of X such that $H \subset G \subset$ s*g-cl(G) ⊂ X − K. Put U = s*g-int(G),V = X − s*g-cl(G). Then U and V are disjoint s*g-open sets of X such that H ⊂ U and K ⊂ V.

3.4. Definition. A function f : X → Y is said to be

1. **s*g- closed** (resp. **gs*g- closed ,πgs*g- closed)** if f (F) is s*g-closed (resp. gs*g-closed ,πgs*g-closed) in Y for every closed set F of X
2. **rc - preserving** [12](resp. **almost closed** [13], **almost s*g- closed, almost gs*g-closed**, **almost πgs*g- closed)** if f (F) is regularly closed (resp. closed, s*g-closed, gs*g- closed, πgs*g- closed) in Y for every F ∈ RC(X).
3. **π-continuous[4]**(resp. **almost π-continuous[4])** if f^{-1}(F) is π-closed in X for every closed (resp. regular closed) set F of Y.
4. **almost πgs*g-continuous** if f^{-1}(F) is πgα*-closed in X for every regular closed set F of Y.

From the definitions stated above, we obtain the following diagram:

closed	⇒	s*g- closed	⇒	gs*g- closed	⇒	πgs*g- closed
⇓		⇓		⇓		⇓
al. closed	⇒	al. s*g-closed	⇒	al. gs*g-closed	⇒	al. πgs*g-closed

where al. = almost.

Where none of the reverse implications are true as can be seen from the following examples :

3.5. Example. X = { a, b, c, d }, τ ={ ϕ , {c}, {a, b, d},X } and σ = { ϕ, {a}, {d}, {c, d}, {a, d},{a, c, d}, X }. Let f : (X, τ) → (X, σ) be the identity function. Then f is πgs*g-closed but not πg-closed. Since A= {c} is not πg-closed in (X, σ).

3.6. Example . Let X = { a, b, c, d } τ = {ϕ, {c}, {b, d}, {a, b, d},{b, c, d}, X } and σ = { ϕ, X, {a}, {d},{a, d}, {c, d}, {a, c, d}}. Let f : (X, τ) → (X, σ) be the identity function Then f is almost πgs*g-closed but not πgs*g- closed. Since A= {c} is not πgs*g-closed.

3.7.Theorem. A surjection f : X → Y is almost πgs*g-closed if and only if for each subset S of Y and each U ∈ RO(X) containing f^{-1}(S), there exists a πgs*g-open set V of Y such that S ⊂ V and f^{-1}(V) ⊂ U.

Proof. Necessity. Suppose that f is almost πgs*g-closed. Let S be a subset of Y and $U \in RO(X)$ containing $f^{-1}(S)$. If $V = Y - f(X - U)$, then V is a πgs*g-open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any regular closed set of X. Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F \in RO(X)$. There exists a πgs*g- open set V of Y such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we have $f(F) \supset Y - V$ and $F \subset X - f^{-1}(V) \subset f^{-1}(Y - V)$. Hence we obtain $f(F) = Y - V$ and f (F) is πgs*g-closed in Y which shows that f is almost πgs*g-closed.

4. Preservation Theorems

4.1.Theorem. If $f : X \rightarrow Y$ is an almost πgs*g-continuous rc-preserving injection and Y is quasi s*g-normal then X is quasi s*g-normal.

Proof. Let A and B be any disjoint π-closed sets of X. Since f is a rc-preserving injection, f (A) and f (B) are disjoint π-closed sets of Y. Since Y is quasi s*g-normal, there exist disjoint s*g-open sets U and V of Y such that $f(A) \subset U$ and $f(B) \subset V$. Now if $G = int(cl(U))$ and $H = int(cl(V))$. Then G and H are regular open sets such that $f(A) \subset G$ and $f(B) \subset H$. Since f is almost πgs*g-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint πgs*g-open sets containing A and B respectively which shows that X is quasi s*g-normal.

4.2.Theorem. If $f : X \rightarrow Y$ is π-continuous almost s*g-closed surjection and X is quasi s*g-normal space then Y is s*g-normal.

Proof. Let A and B be any two disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint π-closed sets of X. Since X is quasi s*g-normal, there exist disjoint s*g-open sets of U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Let $G = int(cl(U))$ and $H = int(cl(V))$. Then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. Set $K = Y - f(X - G)$, $L = Y - f(X - H)$.Then K and L are s*g-open sets of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$, $f^{-1}(L) \subset H$. Since G and H are disjoint, K and L are disjoint. Since K and L are s*g-open and we obtain $A \subset s^*g\text{-}int(K)$, $B \subset s^*g\text{-}int(L)$ and $s^*g\text{-}int(K) \cap s^*g\text{-}int(L) = \phi$. Therefore Y is s*g - normal.

4.3.Theorem. Let f : X →Y be an almost π-continuous and almost πgs*g-closed surjection. If X is quasi s*g-normal space then Y is quasi s*g-normal.

Proof. Let A and B be any disjoint π-closed sets of Y. Since f is almost π-continuous, $f^{-1}(A)$, $f^{-1}(B)$ are disjoint closed subsets of X. Since X is quasi s*g-normal, there exist disjoint s*g-open sets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Put G = int(cl(U)) and H = int(cl(V)).Then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By **Theorem 3.7**, there exist πgs*g-open sets K and L of Y such thatA ⊂ K, B ⊂ L, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint. So are K and L by **Theorem 2.4**, A ⊂ s*g-int(K), B ⊂ s*g-int(L) and s*g-int(K) ∩ s*g-int(L) = ϕ . Therefore, Y is quasi s*g-normal.

4.4.Corollary. If f : X → Y is an almost continuous and almost closed surjection and X is a normal space, then Y is quasi s*g-normal.

Proof. Since every almost closed function is almost πgs*g-closed so Y is quasi s*g-normal.

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