# REDUCIBILITY OF EULERIAN GRAPHS AND DIGRAPHS

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## ABSTRACT:

The concept of digraphs (or directed graphs) is one of the richest theories in graph theory, mainly because of their applications to physical problems. They are applied in abstract representations of computer programs and are an invaluable tools in the study of sequential machines. They are also used for systems analysis in control theory. Most of the concepts and terminology of undirected graphs are also applicable to digraphs, and hence in this chapter more emphasis will be given to those properties of digraphs that are not found in undirected graphs.

## KEYWORDS:

Reducibility of Eulerian Graphs and digraphs , Vertex Reducibility of Eulerian graph's and Digraph, Edge Reducibility of Eulerian graphs and digraphs.

## INTRODUCTION:

The Reducibility of Eulerian Graphs and Digraphs, Vertex Reducibility of Eulerian Graphs and Digraphs, Edge Reducibility of Eulerian Graphs and Digraphs.

**Definition**

Let $\Re$ be a class of graphs satisfying certain property *P*, and $G \in \Re$ . A vertex(edge) *v* in *G* is called deletable with respect to $\Re$ , if $G - v \in \Re$ . In general, a set *S* of Vertices (edges) is called deletable with respect to $\Re$, if $G{-}v{\in}\Re$.

Generally if $S = k.$ then we say that *S* is a k- deletable set.

**Definition**

Let $\Re$ be a class of graphs satisfying a certain property $P$ . The class $\Re$ is called *vertex* (edge) reducible if for any $G \in \Re$ either $\Re$ is the trivial graph (null graph) or it contains a vertex (edge) $v$ such that $G - v \in \Re$ .

***Proposition***

1. The class of trees is vertex reducible, but not edge reducible.
2. The class of connected graphs is vertex reducible.

***Proposition***

1. The class of bipartite graphs is vertex reducible and edge reducible.
2. The class of complete graphs is vertex reducible, but not edge reducible.

***Proposition***

The classes of Hamiltonian graphs, regular graphs, Eulerian graphs are neither edge reducible nor vertex reducible.

**Vertex Reducibility of Eulerian Graphs and Digraphs**

In this section the vertex reducibility number for Eulerian graphs and Eulerian digraphs has been studied. We need the following concept of complementary sub graph.

**Definition Tutte [13]**

Let $H$ be a sub graph of a graph $G$. Then there is a sub graph $H^c$ of $G$ such that $E(H^c) = E(G) - E(H)$ and $V(H^c) = (V(G) - V(H)) \cup W(G, H)$. We call $H^c$ the complementary sub graph to $H$ in $G$.

Firstly, we prove some required lemmas and then using these lemmas to characterize the vertex reducibility number of Eulerian graphs.

**Lemma**

Let $G$ be a graph and $U \subseteq V(G)$. Then the complementary sub graph to $G - U$ is the subgraph whose vertex set is $U \cup N(U)$ and edge set is $\{e \in E(G)$: $e$ is incident with a vertex of $U\}$.

***Proof:***

Let $H$ be the complementary subgraph to $G - U$. By Definition 5.9,

$$V(H) = \{V(G) - V(G - U)\} \cup W(G, G - U) = U \cup W(G, G - U).$$

We have

$$W(G, G - U) = N(U) - U.$$

Hence,

$V(H) = U \cup N(U)$ .

Further,

$E(H) = E(G) - E(G - U)$ .

Let $e \in E(H) = E(G) - E(G - U)$ . We have $e \in E(G)$ and $e \notin E(G - U)$ . This implies that atleast one of the end vertices of $e$ is in $U$. Hence, of $U$ } .

**Theorem**

Let $\mathfrak{I}$ be the class of Eulerian graphs and $G \in \mathfrak{I}$. Then $v - red_{\mathfrak{I}}(G) = k$ , if and only if $k$ is thesmallest number such that there exists a set of vertices $U$ of cardinality $k$ with $H$=$G$–$U$ is connected and $H^c$ is Eulerian.

***Proof:***

Suppose $v - red_{\mathfrak{I}}(G) = k$. There exists $U$, a subset of cardinality $k$ of $V(G)$ such that $H = G - U$ is Eulerian, and $U$ is a smallest such set. Since $H$ is Eulerian we have $H$ is connected. By Lemma 5.12, each vertex in $H^c$ has even degree. To prove Euleriannas of $H^c$, it is enough to prove that $H^c$ is connected.

Suppose $H^c$ is not connected, and $H_1$ is a component of $H$ such that $\phi \neq V(H_1) \cap U = S$ . By Lemma 5.12, $H_1$ is the complementary subgraph to $G - S$. We obtain a contradiction to minimality of $k$ by proving that $G - S$ is Eulerian and $|S| < k$. Note that if S=U then $H_1 = H^c$, a contradiction to our assumption that $H^c$ is not connected. Hence $|S| < k$. Since $H^c$ has no odd vertex, the component $H_1$ has no odd vertex and hence, by Lemma 5.12, $G$–$S$

has no odd vertex. It remains to prove that $G - S$ is connected. If possible, suppose $H_2$ is acomponent of $G - S$ which is disjoint from the component of $G–S$ that contains $H$ . We provethat $H_2$ is detached in $G$. Suppose on the contrary that $e$ is an edge in $G$ with end vertices $x, y$ such that $x \in V(H_2))$ and $y \notin V(H_2)$ . Since $H_2$ is disjoint from $H$ it follows that $y \in S$ . Then, $_c x \in V(H_2) \subseteq U$ and $y \in S \subset V(H_1) \cap U$ and hence $e$ is an edge in $H$. As $H_1$ is a componentof $H^c$, we get that $x, y \in V(H_1)$ and therefore $x \in V(H_1) \cap S$, a contradictionto $x \in V(H_2) \subseteq V(G) - S$ . Thus $H_2$ is detached in $G$ and hence $H_2$ is a proper component of $G$ ,which is impossible. We conclude that $G– S$ is connected.

To prove the smallestness of $k$ , suppose $U_1$ is a set of vertices in $G$ such that $G - U_1$ is connected and the complementary sub graph to i $G–U_1$ is Eulerian. If $|U_1| < k = |U|$ then, as $G - U_1$ is connected and the complementary sub graph to i $G - U_1$ is Eulerian, by Lemma 5.12, $G - U_1$ is Eulerian, a contradiction to $v - red\,\mathfrak{I}\,(G) = k$ .

Conversely, suppose $k$ is the smallest number such that there exists $U \subseteq V(G)$ of cardinality $k$ with $H = G - U$ is connected and $H^c$ is eulerian. By Lemma 5.11, $H$ is Eulerian. Hence, $v - red\,\mathfrak{I}\,(G) \leq k$ . Assume that $v–red\,\mathfrak{I}\,(G) = n < k$ . Let $U_1 \subseteq V(G)$ be a set such that $|U1| = n$ and $G - U_1$ is Eulerian then, as proved in the previous part, we have $G - U_1$ isconnected and the complementary sub graph to $G - U_1$ is Eulerian, which is a contradiction to the choice of $k$ . Hence, $v - red\,\mathfrak{I}\,(G) = k$ .

## Edge Reducibility of Eulerian Graphs and Digraphs

We characterize edge reducibility number of Eulerian graphs.

### Theorem

Let $\Im$ be the class of Eulerian graphs and $G \in \Im$. Then, $e - red_{\Im}(G) = k$ if and only if $k$ is the length of a smallest cycle $C$ in $G$ such that $G - E(C)$ is connected.

***Proof:***

Suppose that $e - red_{\Im}(G) = k$. Then there exists a set of edges $\{e_1, e_2, ..., e_k\}$ such that $G_1 = G - \{e_1, e_2, ..., e_k\}$ is Eulerian. Now, we claim that the edge induced subgraph $C$ of $\{e_1, e_2, ..., e_k\}$ forms a cycle. We consider the following two cases.

**Case 1:** $C$ contains a cycle properly. Let $\{e_1, e_2, ..., e_n\}$ with $n < k$ be a cycle in $C$. We have $G - \{e_1, e_2, ..., e_k\}$ is Eulerian, a contradiction to $e - red_{\Im}(G) = k$.

**Case 2:** $C$ does not contain any cycle. Then $C$ is a forest and has an end vertex. It follows that removal of $C$ from $G$ gives a non Eulerian graph which is a contradiction.

Therefore, $C$ is a cycle. The smallest of $k$ follows immediately.

Conversely, assume that $k$ is the length of a smallest cycle $C$ in $G$ such that $G - E(C)$ isconnected. We prove that $e - red_{\Im}(G) = k$. As $G - E(C)$ is connected, it follows that $G - E(C)$is Eulerian. Hence, $e - red_{\Im}(G) \le k$. If $e - red_{\Im}(G) \ne k$, then there exists an edge set$\{ f, f_2, \ldots, f_n \}$ with $n < k$ such that $G - \{ f_1, f_2, \ldots, f_n \}$ is eulerian. By the previous part of the proof,

the set $\{f, f_2, \ldots, f_n\}$ contains $C_1$ a cycle of smaller length than $k$ such that $G–E(C_1)$ isconnected which is impossible. Hence, $e - red_{\mathfrak{I}}(G) = k$.

**Theorem**

Let $\mathfrak{I}$ be the class of Eulerian digraphs, and $D \in \mathfrak{I}$. Then $a–red_{\mathfrak{I}}(D)=k$ if and only if $k$ is the length of a smallest cycle $C$ in $D$ such that $D - A(C)$ is strongly connected.

***Proof:***

Suppose that $a - red_{\mathfrak{I}}(D) = k$. Then, there exists a set of arcs $\{a_1, a_2, \ldots, a_k\}$ such that$D_1 = D - \{a_1, a_2, \ldots, a_k\}$ is an Eulerian digraph; it is smallest such set. Let $C$ be the sub digraph formed by $\{a_1, a_2, \ldots, a_k\}$. It is clear that $D - A(C)$ is strongly connected. We prove that $C$ is a cycle in $D$. Observe that in-degree and out-degree are equal for every vertex in $C$. In particular, a strong component of $C$ is Eulerian. Hence, if $C$ is not a cyclethen it contains a cycle $C_0$ properly. As $D_1=D–A(C)$ is strongly connected, $D_2=D–A(C_0))$ is also strongly connected, and hence $D_2$ is Eulerian. This contradicts to our assumption that$a - red_{\mathfrak{I}}(D)$ is $k$. Therefore $C$ is a cycle in $D$.

Conversely, assume that $k$ is the length of a smallest cycle $C$ in $D$ such that $D–A(C)$ is strongly connected. We prove that $a - red_{\mathfrak{I}}(D) = k$.

Let $\{a_1, a_2,..., a_k\}$ be the set of arcs of the cycle $C$ . Therefore, $D-\{a_1,a_2,...,a_k\}$ is an Eulerian digraph and we get $a-red\mathfrak{J}(D)\leq k$.

If $a-red\ \mathfrak{J}\ (D)\neq k$, then there exists a set $\{f_1, f_2,..., f_n\}$ of arcs in $D$ , with $n < k$ such that $D-\{f_1, f_2,..., f_n\}$ is an Eulerian digraph.

But then $\{f_1, f_2,..., f_n\}$forms a cycle as proved in the previous part, which is impossible due to the choice of $k$ . We conclude that $a-red\ \mathfrak{J}\ (D)=k$.

**Conclusions**

The concept of digraphs is one of the richest theories in Graph theory mainly because of their applications to physical problems. In this dissertation. We discuss about directed paths and connectedness, Euler digraphs, Hamiltonian digraphs, Matrices of digraphs, reducibility of Eulerian graphs and digraphs.

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