VISCOUS INCOMPRESSIBLE FLUID THROUGH A UNIFORM CIRCULAR PIPE WITH SUCTION FOR UNSTEADY FLOW

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ABSTRACT

In this paper the unsteady flow of viscous incompressible fluid through a uniform pipe of circular cross-section with permeable wall has been studied and the complete solutions for the two velocity distribution have been obtained enclosed form. The axial velocity has been produced and the time dependent linear pressure gradient and the radial outward velocity has been imposed by the porosity on the wall for any value of the section parameter $(\sigma = 0, \sigma \leq 1)$. *It has been found out that the axial velocity is maximum along the axis of the pipe and the velocity profiles are symmetrical about the axis of maximum velocity. The result for velocity distribution has been obtained with suction.*

Keyword: *Viscous incompressible fluid porous wall outward normal, impermeable wall Hagen-Poiseuille flow.*

Introduction

Exact solution of the Navier-Stokes equation has been obtained and it has been found that the velocity distribution over a cross-section takes the form of a paraboloid of revolution. Choudhary R.C. and Sinha K.D.P. obtained the solution of Navier-Stokes equations for the steady motion of a viscous incompressible fluid through a uniform pipe of circular.

Equation of Motion

Let us consider the unsteady flow of viscous incompressible fluid through a uniform circular pipe of circular cross section with permeable wall. Me Navier-Stokes equation of motion in cylindrical co-ordinate for unsteady flow are :

$$\left[\frac{\partial u}{\partial t} + \upsilon \frac{\partial u}{\partial y} + \frac{\omega}{y}\frac{\partial u}{\partial \phi} + u\frac{\partial u}{\partial x}\right]$$

$$= \quad -\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left[\frac{\partial^2 u}{\partial y^2}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{1}{y^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{\partial^2 u}{\partial x^2}\right] \qquad (1)$$

$$\left[\frac{\partial v}{\partial t}+\upsilon\frac{\partial v}{\partial y}+\frac{\omega}{y}\frac{\partial v}{\partial \phi}-\frac{\omega^2}{y}+\upsilon\frac{\partial v}{\partial x}\right]$$

$$= \quad -\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon\left[\frac{\partial^2 v}{\partial y^2}+\frac{1}{y}\frac{\partial v}{\partial y}-\frac{v}{y^2}+\frac{1}{y^2}\frac{\partial^2 v}{\partial \phi^2}-\frac{2}{y}\frac{\partial^2 \omega}{\partial \phi}+\frac{\partial^2 v}{\partial x^2}\right] \qquad (2)$$

$$\left[\frac{\partial w}{\partial t}+\upsilon\frac{\partial \omega}{\partial y}+\frac{\omega}{y}\frac{\partial \omega}{\partial \phi}+\frac{v.\omega}{y}+u\frac{\partial w}{\partial x}\right]$$

$$= \quad -\frac{1}{\rho}\frac{1}{y}\frac{\partial p}{\partial \phi}+\upsilon\left[\frac{\partial^2 \omega}{\partial y^2}+\frac{1}{y}\frac{\partial \omega}{\partial y}-\frac{\omega}{y^2}+\frac{1}{y^2}\frac{\partial^2 \omega}{\partial \phi^2}+\frac{2}{y^2}\frac{\partial v}{\partial \phi}+\frac{\partial^2 \omega}{\partial x^2}\right] \qquad (3)$$

and the equation of continuity is

$$\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{v}{y}+\frac{1}{y}\frac{\partial \omega}{\partial \phi}=0\right] \qquad (4)$$

where u, v and ω denote the axial, radial and azimuthal velocity components is the direction of x, y and z respectively.

Now for laminar flow through a circular pipe $\omega = 0$

$\frac{\partial \omega}{\partial \phi} = 0$, for rotational symmetry

$\frac{\partial u}{\partial x} = 0$, due to axial symmetry

$\frac{\partial v}{\partial t} = 0$, radial velocity being independent of time

Under the above conditions equations (3) becomes identity and equations (1), (2) and (4) reduced to

$$\frac{\partial u}{\partial t}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+v\left[\frac{\partial^2 u}{\partial y^2}+\frac{1}{y}\frac{\partial u}{\partial y}\right] \qquad (5)$$

$$\upsilon\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon\left(\frac{\partial^2 v}{\partial y^2}+\frac{1}{y}\frac{\partial v}{\partial y}-\frac{v}{y^2}\right) \qquad (6)$$

and $$\frac{\partial v}{\partial y}+\frac{v}{y}=0 \qquad (7)$$

with the boundary conditions

$$\left.\begin{array}{l} y = R,\ u = 0\ \ v = V_0 \\ y = 0,\ \dfrac{\partial u}{\partial y} = 0 \end{array}\right] \qquad (8)$$

where R is the radius of the pipe

Now introducing the following dimensionless quantities

$$\left.\begin{array}{l} \bar{x} = \dfrac{x}{R},\ \bar{u} = \dfrac{uR}{\upsilon} \\ \eta = \dfrac{y}{R},\ \bar{\upsilon} = \dfrac{vR}{\upsilon},\ \sigma = \dfrac{V_0 R}{\upsilon} \\ \bar{t} = \dfrac{\upsilon t}{R^2},\ \dfrac{\upsilon}{p} = \dfrac{PR}{\upsilon^2 \rho} \end{array}\right\} \qquad (8a)$$

Equations (5) to (7) be comes

$$\frac{\upsilon}{r}\frac{\partial \bar{u}}{\partial \bar{t}}\frac{\upsilon}{R^2} + \bar{v}\frac{\upsilon}{R}.\frac{\upsilon}{R}\frac{\partial \bar{u}}{\partial \eta}\frac{1}{R} = -\frac{1}{\rho}\frac{\upsilon^2 \rho}{R^2}\frac{\partial \bar{p}}{\partial \bar{x}}\frac{1}{R} + \upsilon\left[\frac{\upsilon}{R^2}.\frac{1}{R}\frac{\partial^2 \bar{u}}{\partial \eta^2} + \frac{1}{\eta R}\frac{\upsilon}{R^2}\frac{\partial \bar{u}}{\partial \eta}\right]$$

$$v\frac{\upsilon}{R}.\frac{\upsilon}{R^2}\frac{\partial \bar{v}}{\partial \eta} = -\frac{1}{\rho}\frac{\upsilon^2}{R^2}\frac{\partial \bar{p}}{\partial \eta}.\frac{1}{R} + \upsilon\left[\frac{\upsilon}{R^2}.\frac{1}{R}\frac{\partial^2 \bar{v}}{\partial \eta^2} + \frac{1}{\eta R}\frac{\upsilon}{R^2}\frac{\partial \bar{v}}{\partial \eta} - \frac{\bar{v}\upsilon}{R\eta^2 R}\right]$$

and $\dfrac{\upsilon}{R}.\dfrac{1}{R}\dfrac{\partial \bar{v}}{\partial \eta} + \dfrac{\upsilon}{\eta}.\dfrac{\bar{v}}{R} = 0$

i.e.

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{v}\frac{\partial \bar{u}}{\partial \eta} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \bar{u}}{\partial \eta} \qquad (9)$$

$$\bar{v}\frac{\partial \bar{v}}{\partial \eta} = -\frac{\partial \bar{p}}{\partial \eta} + \frac{\partial^2 \bar{v}}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \bar{v}}{\partial \eta} - \frac{\bar{v}}{\eta^2} \qquad (10)$$

and

$$\frac{\partial \bar{v}}{\partial \eta} + \frac{\bar{v}}{\eta} = 0 \qquad (11)$$

With boundary conditions

$$\left.\begin{array}{l} \eta = 1, \bar{u} = 0, \bar{v} = \sigma \\ \eta = 0, \dfrac{\partial \bar{u}}{\partial \eta} = 0 \end{array}\right\} \qquad (12)$$

Method of Solution

The solution of equation (11) is given by

$$\log \bar{v}\eta = \text{constant}$$

i.e., $$\bar{v}\eta = \text{k} \tag{13}$$

where k = constant of integration. From (13) and (12) we have

$$\sigma.1 = k \tag{14}$$

From (13) and (14), we have

$$\bar{v}\eta = \sigma$$

$$\bar{v} = \frac{\sigma}{\eta} \tag{15}$$

Equation (10) with the help of (15) becomes

$$\frac{\sigma}{\eta}.\frac{-\sigma}{\eta^2} = -\frac{\partial \bar{p}}{\partial \eta} + \frac{2\sigma}{\eta^3} - \frac{\sigma}{\eta^3} - \frac{\sigma}{\eta^3}$$

i.e., $$-\frac{\sigma^2}{\eta^3} = -\frac{\partial \bar{p}}{\partial \eta}$$

i.e., $$\frac{\partial \bar{p}}{\partial \eta} = \frac{\sigma^2}{\eta^3}$$

Thus $\frac{\partial \bar{p}}{\partial \eta} =$ a function of η only or independent of $\bar{x}$

$$\frac{\partial^2 \bar{p}}{\partial \bar{x} \partial \eta} = 0$$

Or, $$\frac{\partial \bar{p}}{\partial \bar{x}} = \text{Constant or a function of time}$$

Let

$$\frac{-\partial \bar{p}}{\partial \bar{x}} = F(\bar{t}) \tag{16}$$

and $$\bar{u}(\eta, \bar{t}) = \overline{u_0}(\eta) F(\bar{t}) - \overline{u_1}(\eta) F'(\bar{t}) \tag{17}$$

Equation (9) with the help of (16) and (17) becomes

$$\overline{u_1} F'' + \left(\frac{\sigma}{\eta} \bar{u}_1' - \bar{u}_1'' - \frac{1}{\eta} \bar{u}_1' - \bar{u}_0 \right) F'$$

$$+\left(1-\frac{\sigma}{\eta}\bar{u}_0^{'}+\bar{u}_0^{"}+\frac{1}{\eta}\bar{u}_0^{'}\right)F=0 \qquad (18)$$

Thus in order that equation (18) holds, we must have

$$F"\left(\bar{t}\right)=0\ \left(\text{for all } \bar{t}\right) \qquad (18a)$$

and
$$1-\frac{\sigma}{\eta}\bar{u}_0^{'}+\bar{u}_0^{"}+\frac{1}{\eta}\bar{u}_0^{'}=0 \qquad (18b)$$

$$\frac{\sigma}{\eta}\bar{u}_1^{'}-\bar{u}_1^{"}-\frac{1}{\eta}\bar{u}'-\bar{u}_0=0 \qquad (18c)$$

Now the solution of (18a) gives

$$F\left(\bar{t}\right)=A+B\bar{t} \qquad (19)$$

where A and B are dimensional less arbitrary constants.

The equations for $\bar{u}_0$ and $\bar{u}_1$ from (18b) and (18c) are given by

$$\bar{u}_0^{"}+\frac{1-\sigma}{\eta}u+1=0 \qquad (20)$$

and
$$\bar{u}_1^{"}+\frac{1-\sigma}{\eta}\bar{u}_1^{'}+\bar{u}_0=0 \qquad (21)$$

with the boundary conditions

$$\left.\begin{array}{l}\eta=1,\ \bar{u}_0\left(\eta\right)=0,\ \bar{u}_1\left(\eta\right)=0\\ \eta=0,\bar{u}_1^{'}\left(\eta\right)=0,\ \bar{u}_1\left(\eta\right)=0\end{array}\right\} \qquad (22)$$

The solution of the equation (20) satisfying the conditions of (22) is given by

$$\bar{u}_0=\frac{1-\eta^2}{2\left(2-\sigma\right)},0<\sigma\le 1 \qquad (23)$$

Substituting the value of $\bar{u}$ from (23) in (21), we get

$$\bar{u}_1^{"}+\frac{1-\sigma}{\eta}\bar{u}_1^{'}+\frac{1-\eta^2}{2\left(2-\sigma\right)}=0 \qquad (24)$$

Again the solution of the equation (24) satisfying the condition (22) is given by

$$\bar{u}_1=\frac{-\eta^2}{4\left(2-\sigma\right)^2}+\frac{\eta^4}{8\left(2-\sigma\right)\left(4-\sigma\right)}+\frac{6-\sigma}{8\left(2-\sigma\right)^2\left(4-\sigma\right)}\quad 0<\sigma\le 1 \qquad (25)$$

Substituting the values of $\bar{u}_0$ and $\bar{u}_1$ from (23) and (24) in (17), we get

$$\bar{u} = \frac{1-\eta^2}{2(2-\sigma)^2} F(\bar{t}) - F'(\bar{t}) \left[\frac{6-\sigma}{8(2-\sigma)^2(4-\sigma)} - \frac{\eta^2}{4(2-\sigma)^2} + \frac{\eta^4}{8(2-\sigma)(4-\sigma)} \right] \quad (26)$$

Thus the complete solution for the axial velocity is given by (19) and (26) as

$$\bar{u} = (A + B\bar{t}) \left[\frac{1-\eta^2}{2(2-\sigma)} \right] - B \left[\frac{6-\sigma}{8(2-\sigma)^2(4-\sigma)} - \frac{\eta^2}{4(2-\sigma)^2} + \frac{\eta^4}{8(2-\sigma)(4-\sigma)} \right], 0 < \sigma < 1 \quad (27)$$

The discharge of flux per second is

$$q = \int_0^R 2\pi u y dy$$

$$\frac{q}{R\upsilon} = 2\pi \int_0^1 \bar{u}\, \eta\, d\eta$$

$$\bar{q} = 2\pi \left[\frac{A+B\bar{t}}{8(2-\sigma)} - \frac{B}{48} \frac{8-\sigma}{(2-\sigma)^2(4-\sigma)} \right] \quad (28)$$

Particular Result

Now in case of flow through a uniform circular pipe with impermeable wall $(\sigma = 0)$, the axial velocity distribution in (27) reduces to

$$\bar{u} = (A + B\bar{t}) \frac{1-\eta^2}{4} - \frac{B}{64}(1-\eta^2)(3-\eta^2) \quad (29)$$

and the discharge of flux per second in (28) becomes

$$\bar{q} = \frac{A+B\bar{t}}{8}\pi - \frac{\pi B}{48} \quad (29a)$$

The result obtained in (28) and (29) complete agreement with (8) and (9) in case of unsteady flow through a circular tube. For steady case but with porous wall (B = 0, $\sigma \neq 0$; (27) reduces to

$$\bar{u} = \frac{A(1-\eta^2)}{4\left(1-\frac{\sigma}{2}\right)} \quad (30)$$

and (28) reduces to

$$\bar{q} = \frac{\pi A}{4(2-\sigma)} = \frac{\pi A}{8\left(1-\frac{\sigma}{2}\right)} \quad (31)$$

Thus the result obtained in (30) and (31) are in complete agreement with (18) and (22) in case of steady flow through a uniform circular pipe with porous wall.

Again in case of steady flow through a tube of circular cross-section with solid wall $(\sigma = 0)$; equation (30) and (31) reduces to

$$\bar{u} = \frac{A}{4}\left(1-\eta^2\right) \tag{32}$$

and

$$\bar{q} = \frac{\pi A}{8} \tag{33}$$

Thus the results obtained in (32) and (33) completely agree with the well known results for the velocity distribution and the mean velocity of Hagen-Poiseuille flow through a circular pipe with impermeable wall.

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