A M/G/1 QUEUE WITH PREPARATORY WORK ON ARRIVAL CUSTOMERS IN A SINGLE SERVER FEEDBACK CHANNEL

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Abstract

In this paper,we study M/G/1 queue with preparatory work on arriving customers with feedback. The feedback decision process is Bernoulli. The queue length process is studied at departure epoch. The stationary distributions are obtained and the particular cases are derived. The performance measures are also obtained. Finally the numerical example is given to check the correctness.

Keywords:Bernoulli process, Chain, Ergodicity, Embedded, Markov Chain, Feedback, Stationary Distributions.

INTRODUCTION

Queueing theory was introduced by the Danish mathematician Erlang. The Erlang work [1] on queueing stimulated many authors to develop a variety of queueing models. A queue is nothing but a waiting line in which customers wait for receiving a service. In certain queueing models, before starting a service, the server may have to do some preparatory work or some alignment must be done in the case of certain necessities. This sort of preparatory work for customers occur in Hospitals,Production Processes,Banks etc.Krishna Reddy,Nadarajan and Arumuganathan [4] have studied an M/*G(a,b)/1 queue with N-policy multiple vacation and setup times.Takac [8] introduced queues with feedback mechanism. The queueing systems which include the possibility for a customer to return to the service counter for additional service are called queues with feedback. Santhakumaran and Thangaraj [5]have proposed a single server queue with

impatient and feedback customers.Santhakumaran and Shanmugasundaram [6] have studied preparatory work on arriving customers with a single server feedback queue .Santhakumaran ,Ramasamy and Shanmugasundaram[7] have also studied a single queue with instantaneous Bernoulli feedback and preparatory work. Gautam Choudhury and Chandapaul [3] have studied an M/G/1 queue with two phase of heterogeneous services and Bernoulli feedback system where the server provides first phase of regular service to all the customers. Disney Menickle and

Simon [2] have studied several random processes that occur in M/G/1 queue with instantaneous Bernoulli feedback.

Description of the Model

In this paper, we are concerned with the random processes in which the customers entering into the queue is served with a preparatory work and then move to the service counter for service

.The preparatory work and service times are independent but not identically distributed. The stationary distributions of the departure process is derived. Some particular cases are deduced and performance measures are obtained.

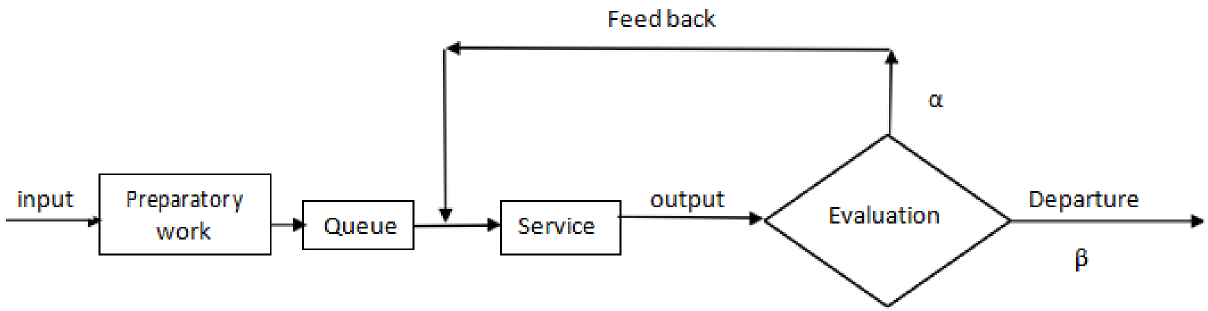


Figure 1. Flow of customers

The above figure illustrates the flow of customers through the queueing system. The arrival process is considered to be a poisson process in this system and a preparatory work is done for each customer prior to getting service. After getting the service depending on the level of service, a decision is made whether to depart or feedback. Let β be the probability for departure and α be the probability of feedback customers such that $\alpha + \beta = 1$. The capacity of the queue is infinite and the queue discipline is First Come First Served.

Notations

In this system, the customers entering into the queue is served with a preparatory work and then move to the service counter for service. The queueing system is denoted by M/G/1 queue with preparatory work on arriving customers and with instantaneous Bernoulli feedback. The arrival process is a poisson distribution with parameter $\lambda>0$. Preparatory and service time follow general distribution with parameters μ_1 and μ_2 respectively. Let S_1 be the preparatory work time and S_2 be the service time.

The random variables S_1 and S_2 are independent and identically distributed non negative random variables with $E(S_1)<\infty$ and $E(S_2)<\infty$ with the pdf $H(t) = P(S_1+S_2<\infty) = \int_0^t f_1(u)\, f_2(t-u)du = f_1(t) * f_2(t)$.

The arrival process, preparatory work and service times are independent Processes. The preparatory work time epochs at $X_1', X_2', X_3' \ldots\ldots X_n'$ and service time epochs occur at $Y_1', Y_2', Y_3', \ldots\ldots Y_n'$, service completions occur at $D_1<D_2<\ldots\ldots<D_n$ called the output epochs,where $D_i = X_i' \cup Y_i'$, $i = 1,2\ldots\ldots n$.

Let V_n be the random variable denoting departure or feedback after getting service i.e., $V_n = V(t_n) = \begin{cases} 0 & \text{if the } n^{th} \text{output departs.} \\ 1 & \text{if the } n^{th} \text{output feedsback.} \end{cases}$

Here $\{V_n\}$ is a Bernoulli process. Elements of the subset $\{t_n\} \subset \{D_n\}$ are called departure epochs and the times at which an output leaves the system. The elements of the subset $\{\tau_n\} \subset \{D_n\}$ are called the feedback epochs and $t_1, t_2, t_3, \ldots\ldots\ldots. t_n$ are the times at which the output returns to the queue such that $\{t_n\} \cup \{\tau_n\} = \{D_n\}$. The time D_n' are the times at which a unit enters the queue, $\{D_n\}$ are called input epochs such that $\{W_n\} \cup \{\tau_n\} = \{D_n'\}$. Let N(t) be the queue length at time then $N_1^+(n) = N(W_n - 0)$ and $N_2^+(n) = N(D_n')$, $N_3^+(n) = N(t_n - 0)$ and $N_4^+(n) = N(t_n + 0)$ are respectively the embedded queue lengths at arrival, input, output and departure epochs.

Queue length at Departure Epochs

The stationary probability distribution of the departure process $\{N_4^+(n)\}$ is discussed in this section. Let $(S_{11}\delta_{11}+S_{21}) + (S_{12}\delta_{12}+S_{22}) + (S_{13}\delta_{13}+S_{23}) + \ldots\ldots + (S_{1k}\delta_{1k}+S_{2k})$ be the total service time and the preparatory work between $(n - 1)^{th}$ and n^{th} departure, where S_{1k} is the preparatory work and S_{2k} is the service time, k is the number of services performed between the $(n - 1)^{th}$ and n^{th} departure and δ_{ik} is the kroneckor number

$$D_n = \begin{cases} D_{n-1} + s_n{}' \text{ if } N_4^+(n-1) > 0 \\ D_{n-1} + I_n + s_n{}' \text{ if } N^+(n-1) = 0 \end{cases}$$

where I_n is the idle time following D_{n-1} when $N_4^+(n-1) = 0$. Without loss of generality, it is assumed that $\{ S_n : n = 2,3,\ldots\ldots\}$ is a sequence of independent identically distributed random variables.

Lemma 1. The process $\{N_4^+(n-1); D_n - D_{n-1}\}$ is the Markov Renewal process with

kernel

$$A(i,j,x) = P\{N_4^+(n) = j;\ D_n - D_{n-1} \le x \,/\, N_4^+(n-1) = i\},$$

Define

$$P_j(y) = \frac{e^{-\lambda y} (\lambda y)^j}{j!}, \quad j = 1,2,3,\ldots\ldots$$

$$A(i,j,x) = \begin{cases} 0 & , \quad \text{if } j < (i-1) \\ \int_0^x P_{j-i+1}(y)\, dH(y) & , \quad \text{if } i \neq 0, j \geq i-1, \\ \int_0^x (1 - e^{-\lambda(x-y)}) P_j(y) dH(y) & , \quad \text{if } i = 0, j > 0, \\ \int_0^x (1 - e^{-\lambda(x-y)}) P_0(y) dH(y) & , \quad \text{if } i = 0, j = 0 \end{cases}$$

Proof. We have

$$D_n = \begin{cases} D_{n-1} + s_{n'}^{\,'} & \text{if } N_4^+(n-1) > 0 \\ D_{n-1} + I_n + s_{n'}^{\,'} & \text{if } N_4^+(n-1) = 0 \end{cases}$$

where I_n is exponentially distributed idle time preceding $s_{n'}^{\,'}$ when $N_4^+(n-1) = 0$.Using standard embedded Markov chain technique, the above result can be easily

Proved.

Lemma 2. The Laplace stieltje's transform $G^*(s)$ of the distribution function of $s_{n'}^{\,'}$ is given by

$$G^*(s) = \frac{\beta H^*(s)}{1 - \alpha H^*(s)}$$

Proof.

The probability mass function of the number of feedbacks k is

$$P(K=k) = \begin{cases} \beta\alpha^{k-1}, & k=1,2,3, \ldots\ldots \\ 0, & \text{otherwise} \end{cases}$$

The Laplace stieltje's transform of the distribution $s_n^{\,'}$ is given by

$G^*(s) = \sum_{k=1}^{\infty} \beta\alpha^{k-1} [H^*(s)]^{k-1}$

where $H^*(s)$ is the Laplace stieltje's transform of H(t).

$$G^*(s) = \frac{\beta H^*(s)}{1 - \alpha H^*(s)} \quad , s > 0$$

Theorem If $\frac{\lambda[E(s_1)+E(s_2)]}{\beta}$<1, then the stationary distribution of the queue

length at departure epochs is $\pi^d(0), \pi^d(1), \pi^d(2), \ldots\ldots\ldots\ldots$ and its probability

generating function is given by

$$\varphi(z) = \frac{\pi_0^d(z-1)G^*(\lambda-\lambda z)}{z-G^*(\lambda-\lambda z)}, \ |z|<1$$

where $\varphi(z) = \sum_0^\infty \pi_j^d z^j$ and $\pi_j^d = 1- \frac{\lambda[E(s_1)+E(s_2)]}{\beta}$

Proof. In Lemma 1, as $x \to \infty$, we get A(i,j,x)→ A(i,j) which is the (i,j)th element of one step transition probability matrix for $\{N_4^+(n)\}$ process,

$$\text{i.e.,} \qquad A(i,j) = \begin{cases} K_{j-1}, & \text{if } j \geq (i-1) \\ 0, & \text{if } j < (i-1) \end{cases}$$

where $K_j = \int_0^\infty \frac{e^{-\lambda y}(\lambda y)^j}{j!} dH(y)$, $\qquad j = 0,1,2,\ldots\ldots\ldots$

It is known that $\pi^d P = \pi^d$ where P = A(i,j) and $\pi^d = (\pi_0^d, \pi_1^d, \pi_2^d, \ldots\ldots.)$.

In general $\pi_i^d = \pi_0^d K_i + \sum_{j=1}^{i+1} \pi_j^d K_{i-j+1}$, $\qquad$ i=0,1,2,……..

Define the generating function

$$K(z) = K_j z^j \text{ , } |z| \leq 1 \text{ and } \varphi(z) = \sum_{i=0}^\infty \pi_j^d z^i$$

$$\varphi(z) = \sum_{i=0}^\infty [\pi_0^d K_i + \sum_{j=1}^{i+1} \pi_j^d K_{i-j+1}] z^i$$

$$= \pi_0^d \sum_{i=0}^\infty K_i z^i + \sum_{i=0}^\infty \sum_{j=1}^{i+1} \pi_j^d K_{i-j+1} z^i = \frac{\pi_0^d(z-1)K(z)}{z-K(z)}$$

Also $\qquad K(z) = \sum_{j=0}^\infty K_j z^j = \sum_{j=0}^\infty \{ \int_0^\infty \frac{e^{-\lambda y}(\lambda y)^j}{j!} dH(y) \} z^j$

$$= \int_0^\infty \frac{e^{-\lambda y}(\lambda y z)^j}{j!} dH(y) = \int_0^\infty e^{-\lambda y} e^{\lambda y z} \, dH(y)$$

$$= \int_0^\infty e^{-(\lambda-\lambda z)y} \, dH(y) = G^*(\lambda-\lambda z)$$

π_j^d can be obtained from the relation

$$\pi_{n-1}^d = \sum_{r=0}^{n-1} \pi_{n-1}^d K_r + \pi_0^d K_{n-1} , \qquad n=1,2,3,\ldots\ldots\ldots\ldots$$

In order to find π_0^d, allow z→1 in $\varphi(z)$,

$$\lim_{z\to 1}\varphi(z) = \lim_{z\to 1}\frac{\pi_0^d (z-1)\, G^*(\lambda-\lambda z)}{(1-G^*(\lambda-\lambda z))}$$

$$\pi_0^d = 1+\lambda G^{*\prime}(0)$$

$$G^{*\prime}(0) = \frac{H^{*\prime}(0)}{1-\alpha}$$

where $H^*(0) = F_1'(0)+F_2'(0)$ with $F_1'(0) = -E(S_1)$ and $F_2'(0) = -E(S_2)$

$G^{*\prime}(0)$ can be obtained as $G^{*\prime}(0) = \frac{-E(S_1)-E(S_2)}{\beta}$ and

$\pi_0^d = 1+\lambda G^{*\prime}(0) = 1- \frac{\lambda[E(s_1)+E(s_2)]}{\beta}$

Particular case of departure epochs

If $f_1(t)=\mu_1 e^{-\mu_1 t}$ and $f_2(t) =\mu_2 e^{-\mu_2 t}$

i.e., $E[S_1] = \frac{1}{\mu_1}$ and $E[S_2] = \frac{1}{\mu_2}$, then

$$\pi_0^d = 1- \frac{\lambda[E(s_1)+E(s_2)]}{\beta} = 1 - \frac{\lambda}{\beta}[\frac{1}{\mu_1} + \frac{1}{\mu_2}]$$

$$\pi_1^d = \pi_0^d K_1$$

$$H_1(t) = f_1(t) * f_2(t) = \frac{(\mu_1\mu_2)}{(\mu_2-\mu_1)}(e^{-\mu_1 t} - e^{-\mu_2 t})$$

$$\pi_1^d = \left[1 - \frac{\lambda}{q}(\frac{1}{\mu_1} + \frac{1}{\mu_2})\right](\mu_1\mu_2)\,\lambda\left(\frac{\mu_1+\mu_2+2\lambda}{(\lambda+\mu_1)^2(\lambda+\mu_2)^2}\right)$$

Also, when $\frac{1}{\mu_2} \to 0$

$$\pi_1^d=(1 - \frac{\lambda}{\beta\mu_1})(\frac{\lambda}{\mu_1})^j, \qquad j = 0,1,2,\ldots\ldots\ldots\ldots$$

which gives the stationary distribution of the queue in the classical (M/M/1) : (∞/FCFS) model.

Operating Characteristics

The following results are obtained by making use of the probability generating function and by simple straight forward calculations

(i) If N denote the number of customers in the system with instantaneous Bernoulli Processes and set up time , the average number of customers in the system is given by

$$E(N) = \frac{[(\beta A_2 - \lambda\beta A_1)(2\alpha A_2 + \lambda\beta A_1 + 2\lambda\alpha A_1)] + [\lambda^2\beta A_1^2 - 2\lambda\alpha A_1 A_2 - \lambda^2\alpha(\mu_1^2 + \mu_2^2)]}{A_2[\beta A_2 - \lambda\beta A_1 - \lambda\mu_2]}$$

Where $A_1 = (\mu_1 + \mu_2)$ and $A_2 = \mu_1\mu_2$

(ii) If $\frac{1}{\mu_1} \to 0$ (no set up time) and $\beta = 1$ (no feedback), then

$$E(N) = \frac{\frac{\lambda}{\mu_2}}{1 - \frac{\lambda}{\mu_2}}$$

The result exactly coincides with ordinary (classical) queue.

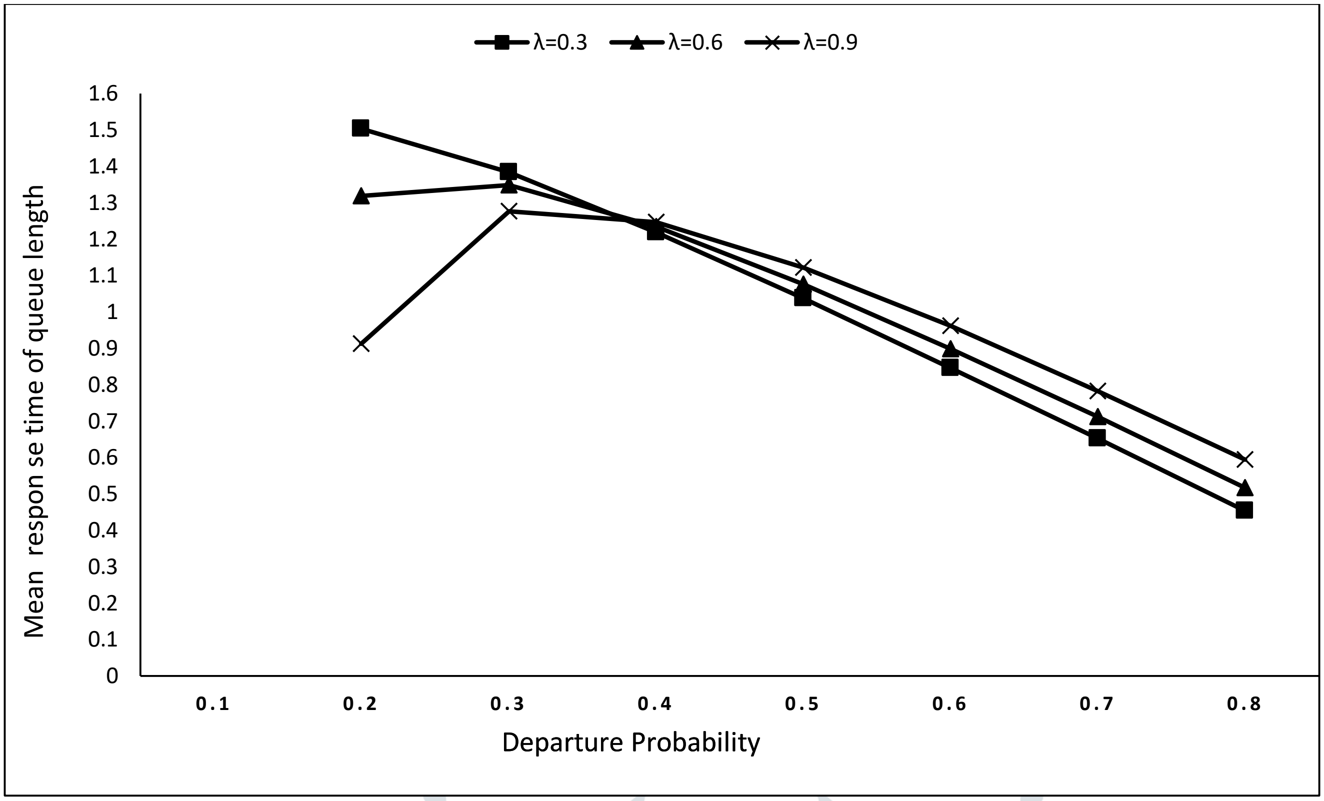
Numerical Study

Based on the mean response time of the model, a numerical study is made. The influence of the parameters β and λ on the mean response time is illustrated. The setup or preparatory work, service rates are fixed and the values of the departure probability is varied. The table is computed on the basis of the condition $\mu_1 < \mu_2$ for various values of β and λ.

Table

Mean Response time of the system E(N) is computed for $\mu_1 = 10$ and $\mu_2 = 15$

μ_1=10 μ_2=15			
β	λ=0.3	λ=0.6	λ=0.9
0.2	1.5035	1.318667	0.913
0.3	1.384569	1.348857	1.2762727
0.4	1.218914	1.234933	1.24632
0.5	1.036461	1.076667	1.1224627
0.6	0.846074	0.899	0.9617143
0.7	0.651315	0.711123	0.782703
0.8	0.45389	0.517212	0.593322



Conclusion

We analyze M/G/1 queueing model at departure epoch with feedback and Preparatory work for customers. The probability distributions under embedded markov chains are derived. Some particular cases are deduced. The performance measures are analyzed under this queueing model. In numerical study, the graph shows that the increase in departure probability decreases the queue length under various arrival rate and also the mean response time of queue length decreases with departure probability under the same situation. It shows the correctness of the model.

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