AN APPLICATION TO HEALTH CARE SYSTEM IN (M/M/1) QUEUEING MODEL

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Abstract: In this paper, we study a queueing model that applies to health care system. It describes how the health system behaves as the number of customers (patients) increases. We consider the hospital system (clinic), here the patients (couples) arrival and services are Markovians . The couples directly enter in to the first node of the system, after completing the service at the first node the couples having two options with probability p (enter in to the second node) and 1-p (enter in to the third node) respectively. After completing the service at the second node the couples can either leave the system or enter in to the third node with probability q (leave the system) and 1-q (enter in to the third node) respectively. After getting the service in the third node the couples leave the system. We formulate this in to a queueing system and obtain the steady state probabilities. Some performance measures are derived, a numerical examples are given to test the feasibility.

Keywords: Queueing-Performance measures- Steady state probabilities.

I INTRODUCTION

There are many situations in daily life where a queue is formed. Machines waiting to be repaired, Patients waiting in a doctor's room, Cars waiting at a traffic signal, we wait in line at banks and post offices etc. There is more demand for service than there is facility for service available .The reasons are there may be (i) Shortage of available servers (ii) Space limit to the amount of service that can be provided. Long queues may result in lost sales and lost customers. The problem of interest is how to achieve a balance between the costs associated with the prevention of waiting in order to maximize the profits. As Queueing theory provides an answer to this problem, it has become a topic of interest. Queue is nothing but a waiting line which is invented by the Danish Mathematician A.K. Erlang in 1909.Erlang [3] has explained the theory of probabilities and telephone conversations. Gross and Harris [4] have explained the fundamentals of Queueing theory. Bose [2] has proposed an introduction to Queueing theory.

Queue network can be regarded as a group of interconnected nodes, where each node represents a service facility of some kind with servers at each node. The Queueing networks were first identified by James. R. Jackson in 1957. An earlier product- form solution was found by Jackson [5] for tandem queues. Jackson [6] has also explained the network of waiting lines. The most significant contribution in queueing network is Jackson's network. Queueing network models have various applications in many areas, such as service centers, computer networks, communication networks, production and flexible manufacturing systems, airport terminals and healthcare systems etc. Queueing networks can be classified as open, closed and mixed networks. In an open network customers enter from outside, receive service at systems and leave the network. In closed network new customers never enter in to and the existing customers never depart from the system. In mixed network, the network may be open for some classes of customers and closed for some other classes.

Patients in hospitals have been extensively studied by many researchers. Avishai Mandelbaum et al. [1] have discussed about the data driven appointment –scheduling under uncertainty in a cancer unit. Ravikant Patel and Hinaben R Patel [9] had analyzed the waiting time and outpatient satisfaction. Stefan Creemers and Marc

R.Lambrecht [11] have presented a study of waiting time in the orthopedic department. Sreekala and Manoharan [10] have focused on a queueing network model with feedback and its application in healthcare. Mor Armony et al. [8] have investigated on patient flow in hospitals. Jackson, Welch and Fry [7] have discussed about the appointment systems in hospitals and general practice.

In this paper we consider an infertility clinic as an open queuing network system . In healthcare systems servers correspond to specialized physicians or equipments and customers to patients.

II DESCRIPTION OF THE MODEL

We consider an open queueing network consisting of three single nodes. The patients arrive to the system according to a Poisson process and get service with a general service time distribution. The patients (couples) first enter in to the first node with rate λ. If the couples need counseling they routed to the second node with probability p. If they need treatment they directed to the third node with probability $1-p$. After the counseling, the couples who realized that they don't require treatment leave the system with probability q. If they want treatment, they directed to the third node with probability $1-q$. Each node follows an M/M/1 schedule. The service rates for node 1, node 2 and node 3 are exponentially distributed with service rates μ_1, μ_2, and μ_3 respectively. Figure 1 represents the system.

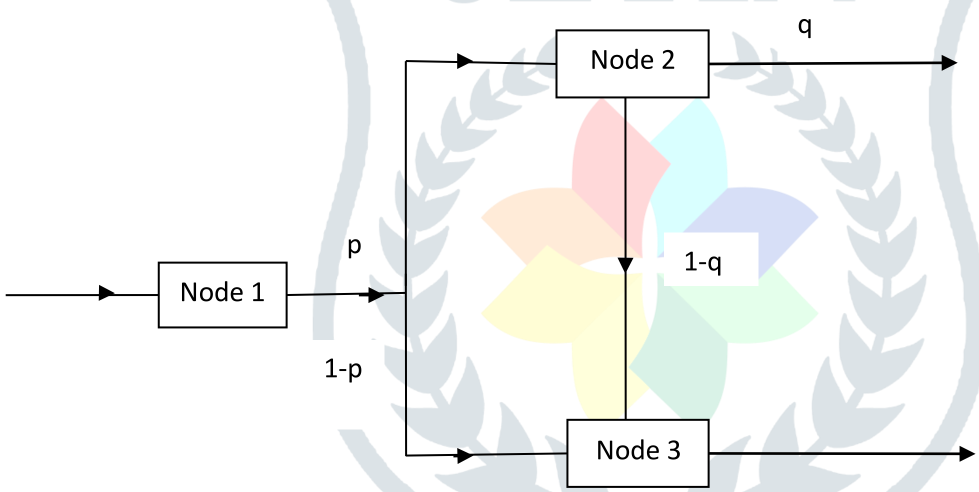

Figure 1

III BALANCE EQUATIONS

We define λ_i where (i=1, 2, 3) is the arrival rate to each nodes. The balance equations for this model are obtained as given below:

$$\lambda_1 = \lambda \quad (1)$$

$$\lambda_2 = \lambda p \quad (2)$$

$$\lambda_3 = \lambda(1-p) + \lambda_2(1-q) \quad (3)$$

$$\lambda_3 = \lambda(1-pq) \quad (4)$$

If n_1, n_2, n_3 are the number of customers at each nodes then using Jackson network the steady state solution is denoted by $P(n_1, n_2, n_3)$.

The steady state probability for n_1, n_2, n_3 customers at the three nodes respectively is

$$P(n_1, n_2, n_3) = (1-\rho_1)\rho_1^{n_1}(1-\rho_2)\rho_2^{n_2}(1-\rho_3)\rho_3^{n_3} \quad (5)$$

Where, $\rho_1 = \frac{\lambda_1}{\mu_1}$, $\rho_2 = \frac{\lambda_2}{\mu_2}$, $\rho_3 = \frac{\lambda_3}{\mu_3}$

$$P(n_1, n_2, n_3) = \left(1-\frac{\lambda_1}{\mu_1}\right)\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left(1-\frac{\lambda_2}{\mu_2}\right)\left(\frac{\lambda_2}{\mu_2}\right)^{n_2}\left(1-\frac{\lambda_3}{\mu_3}\right)\left(\frac{\lambda_3}{\mu_3}\right)^{n_3}.$$

$$= \left(1-\frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_1}\right)^{n_1}\left(1-\frac{\lambda p}{\mu_2}\right)\left(\frac{\lambda p}{\mu_2}\right)^{n_2}\left(1-\frac{\lambda(1-pq)}{\mu_3}\right)\left(\frac{\lambda(1-pq)}{\mu_3}\right)^{n_3} \quad (6)$$

Average number of customers in the system:

Let N_i $(i = 1,2,3)$ be the number of couples in the node i .

We have $N_1 = \frac{\rho_1}{1-\rho_1}$

$$= \frac{\lambda}{\mu_1 - \lambda} \quad (7)$$

$$N_2 = \frac{\rho_2}{1-\rho_2}$$

$$= \frac{\lambda_2}{\mu_2 - \lambda_2}$$

$$= \frac{\lambda p}{\mu_2 - \lambda p} \quad (8)$$

$$N_3 = \frac{\rho_3}{1-\rho_3}$$

$$= \frac{\lambda_3}{\mu_3 - \lambda_3}$$

$$= \frac{\lambda(1-pq)}{\mu_3 - \lambda(1-pq)} \quad (9)$$

Average number of couples in the overall system,

$$L_s = N_1 + N_2 + N_3 \quad (10)$$

$$= \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda p}{\mu_2 - \lambda p} + \frac{\lambda(1-pq)}{\mu_3 - \lambda(1-pq)} \quad (11)$$

Average waiting time of a couple in the system,

$$W_s = \frac{L_s}{\lambda} \quad (12)$$

$$= \frac{1}{\mu_1 - \lambda} + \frac{p}{\mu_2 - \lambda p} + \frac{(1-pq)}{\mu_3 - \lambda(1-pq)} \quad (13)$$

Average number of couples in the queue

$$L_q = L_s - \frac{\lambda}{\mu} \quad (14)$$

$$= \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda p}{\mu_2 - \lambda p} + \frac{\lambda(1-pq)}{\mu_3 - \lambda(1-pq)} - \frac{\lambda}{\mu} \quad (15)$$

Average waiting time of a couple in the queue

$$W_q = \frac{L_q}{\lambda} \quad (16)$$

$$= \frac{1}{\mu_1-\lambda} + \frac{p}{\mu_2-\lambda p} + \frac{(1-pq)}{\mu_3-\lambda(1-pq)} - \frac{1}{\mu} \quad (17)$$

IV NUMERICAL EXAMPLES

In this section we investigate the steady state solution and the performance measures for two set of values. For one set:

For $\lambda = 0.3,\ \mu_1 = 1.2,\ \mu_2$=2.3, $\mu_3 = 3.1, p = 0.3, q = 0.5,$

Average number of couples in the overall system $L_s = 0.4636$

Average waiting time of a couple in the system $W_s = 1.5453$

Average number of couples in the queue $L_q = 0.3272$

Average waiting time of a couple in the queue $W_q = 1.0907$

The steady state probability values for n_1, n_2, n_3 customers at the three nodes respectively are given in Table 1

Table 1

(n_1,n_2,n_3)	$P(n_1,n_2,n_3)$	(n_1,n_2,n_3)	$P(n_1,n_2,n_3)$	(n_1,n_2,n_3)	$P(n_1,n_2,n_3)$	(n_1,n_2,n_3)	$P(n_1,n_2,n_3)$
0 0 0	6.61E-01	1 0 0	1.65E-01	2 0 0	4.13E-02	3 0 0	1.03E-02
0 0 1	5.44E-02	1 0 1	1.36E-02	2 0 1	3.40E-03	3 0 1	8.50E-04
0 0 2	4.48E-03	1 0 2	1.12E-03	2 0 2	2.80E-04	3 0 2	6.99E-05
0 0 3	3.68E-04	1 0 3	9.20E-05	2 0 3	2.30E-05	3 0 3	5.75E-06
0 1 0	2.59E-02	1 1 0	6.47E-03	2 1 0	1.62E-03	3 1 0	4.04E-04
0 1 1	2.13E-03	1 1 1	5.32E-04	2 1 1	1.33E-04	3 1 1	3.33E-05
0 1 2	1.75E-04	1 1 2	4.38E-05	2 1 2	1.09E-05	3 1 2	2.74E-06
0 1 3	1.44E-05	1 1 3	3.60E-06	2 1 3	9.00E-07	3 1 3	2.25E-07
0 2 0	1.01E-03	1 2 0	2.53E-04	2 2 0	6.33E-05	3 2 0	1.58E-05
0 2 1	8.33E-05	1 2 1	2.08E-05	2 2 1	5.21E-06	3 2 1	1.30E-06
0 2 2	6.85E-06	1 2 2	1.71E-06	2 2 2	4.28E-07	3 2 2	1.07E-07
0 2 3	5.64E-07	1 2 3	1.41E-07	2 2 3	3.52E-08	3 2 3	8.81E-09
0 3 0	3.96E-05	1 3 0	9.91E-06	2 3 0	2.48E-06	3 3 0	6.19E-07
0 3 1	3.26E-06	1 3 1	8.15E-07	2 3 1	2.04E-07	3 3 1	5.09E-08
0 3 2	2.68E-07	1 3 2	6.70E-08	2 3 2	1.68E-08	3 3 2	4.19E-09
0 3 3	2.21E-08	1 3 3	5.51E-09	2 3 3	1.38E-09	3 3 3	3.40E-10

For the arrival rate λ from 0.3 to 0.7 and service rate from 1.2 to 1.6 the average number of customers and the average waiting time of couples are calculated in Table 2 and Table 3. From Figure 2 and Figure 3 it is clear that as the arrival rate increases the number of customers and the waiting time increases.

Table 2

λ/μ_1	1.2	1.3	1.4	1.5	1.6
0.3	0.4636	0.4303	0.4030	0.3803	0.3611
0.4	0.6782	0.6227	0.5782	0.5418	0.5115
0.5	0.9430	0.8537	0.7843	0.7287	0.6832
0.6	1.2818	1.1389	1.0318	0.9485	0.8818
0.7	1.7380	1.5047	1.338	1.2130	1.1158

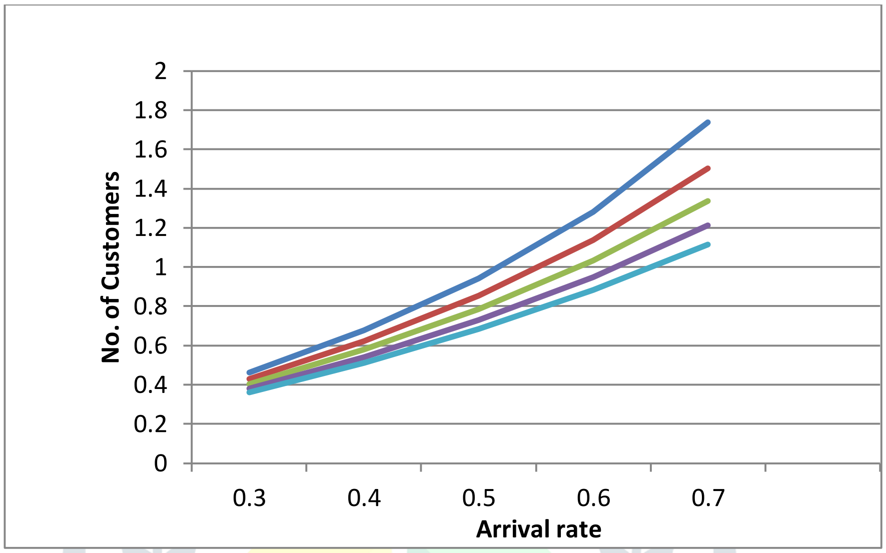

Figure 2

Table 3

λ/μ_1	1.2	1.3	1.4	1.5	1.6
0.3	1.5453	1.4343	1.3433	1.2677	1.2037
0.4	1.6955	1.5578	1.4455	1.3545	1.2788
0.5	1.8860	1.7074	1.5686	1.4574	1.3664
0.6	2.1363	1.8982	1.7197	1.5808	1.4697
0.7	2.4829	2.1496	1.9114	1.7329	1.5940

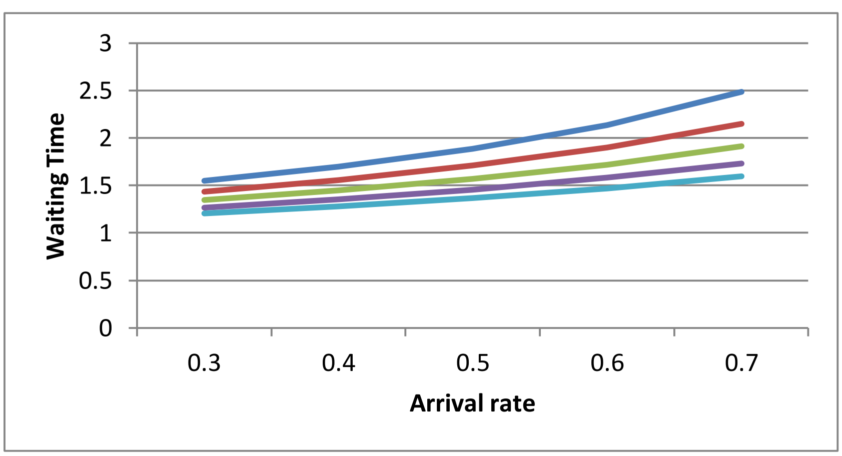

Figure 3

For the service rate from 1.2 to 1.6 and the arrival rate λ from 0.3 to 0.7 the average number of customers and the average waiting time of couples are calculated in Table 4 and Table 5. From Figure 4 and Figure 5 it is clear that as the service rate increases the number of customers and the waiting time decreases.

Table 4

μ_1/λ	0.3	0.4	0.5	0.6	0.7
1.2	0.4636	0.6782	0.9430	1.2818	1.7380
1.3	0.4303	0.6227	0.8537	1.1389	1.5047
1.4	0.4030	0.5782	0.7843	1.0318	1.3380
1.5	0.3803	0.5418	0.7287	0.9485	1.2130
1.6	0.3611	0.5115	0.6832	0.8818	1.1158

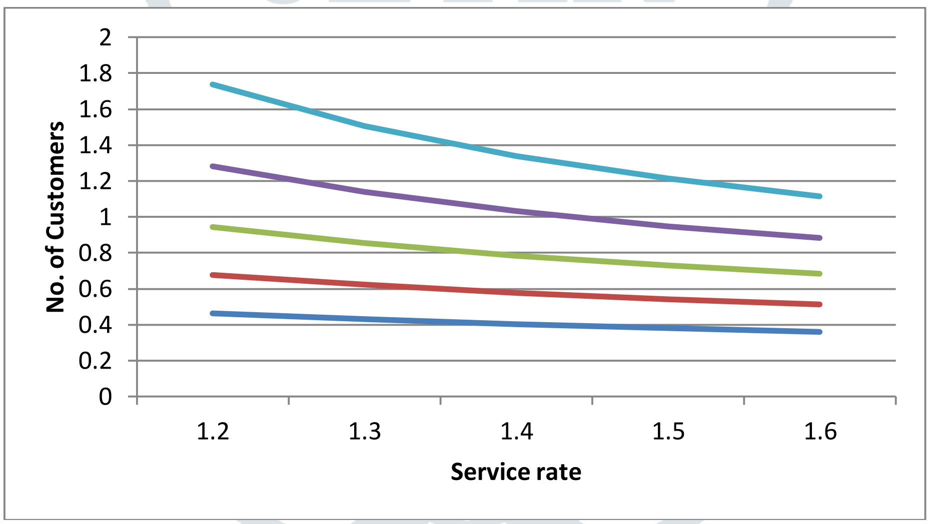

Figure 4

Table 5

μ_1/λ	0.3	0.4	0.5	0.6	0.7
1.2	1.5453	1.6955	1.886	2.1363	2.4829
1.3	1.4343	1.5578	1.7074	1.8982	2.1496
1.4	1.3433	1.4455	1.5686	1.7197	1.9114
1.5	1.2677	1.3545	1.4574	1.5808	1.7329
1.6	1.2037	1.2788	1.3664	1.4697	1.5940

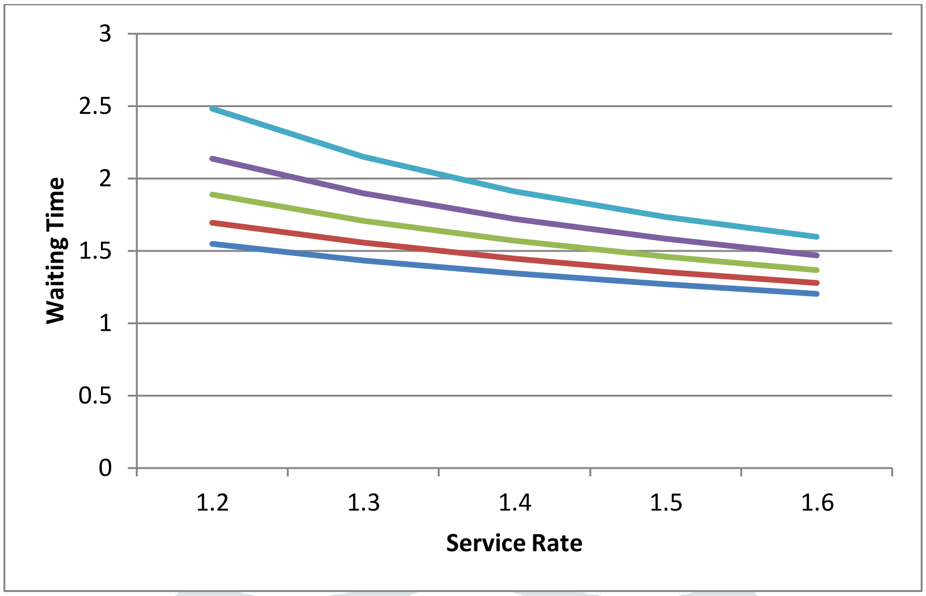

Figure 5

For another set of values:

$\lambda = 0.01,\ \mu_1 = 0.1,\ \mu_2$=0.2, $\mu_3 = 0.3, p = 0.03, q = 0.05$

Average number of couples in the overall system $L_s = 0.14$

Average waiting time of a couple in the system $W_s = 14.7$

Average number of couples in the queue $L_q = 0.09$

Average waiting time of a couple in the queue $W_q = 9$

The steady state probability values for n_1, n_2, n_3 customers at the three nodes respectively are given in Table 6

Table 6

(n_1,n_2,n_3)	$P(n_1,n_2,n_3)$	(n_1,n_2,n_3)	$P(n_1,n_2,n_3)$	(n_1,n_2,n_3)	$P(n_1,n_2,n_3)$	(n_1,n_2,n_3)	$P(n_1,n_2,n_3)$
0 0 0	8.69E-01	1 0 0	8.87E-02	2 0 0	8.69E-03	3 0 0	8.69E-04
0 0 1	2.89E-02	1 0 1	2.89E-03	2 0 1	2.89E-04	3 0 1	2.89E-05
0 0 2	9.62E-04	1 0 2	9.62E-05	2 0 2	9.62E-06	3 0 2	9.62E-07
0 0 3	3.20E-05	1 0 3	3.20E-06	2 0 3	3.20E-07	3 0 3	3.20E-08
0 1 0	1.30E-03	1 1 0	1.30E-04	2 1 0	1.30E-05	3 1 0	1.30E-06
0 1 1	4.34E-05	1 1 1	4.34E-06	2 1 1	4.34E-07	3 1 1	4.34E-08
0 1 2	1.44E-06	1 1 2	1.44E-07	2 1 2	1.44E-08	3 1 2	1.44E-09
0 1 3	4.80E-08	1 1 3	4.80E-09	2 1 3	4.80E-10	3 1 3	4.80E-11
0 2 0	1.95E-06	1 2 0	1.95E-07	2 2 0	1.95E-08	3 2 0	1.95E-09
0 2 1	6.50E-08	1 2 1	6.51E-09	2 2 1	6.51E-10	3 2 1	6.51E-11
0 2 2	2.17E-09	1 2 2	2.17E-10	2 2 2	2.17E-11	3 2 2	2.17E-12
0 2 3	7.21E-11	1 2 3	7.21E-12	2 2 3	7.21E-13	3 2 3	7.21E-14
0 3 0	2.93E-09	1 3 0	2.93E-10	2 3 0	2.93E-11	3 3 0	2.93E-12
0 3 1	9.76E-11	1 3 1	9.56E-12	2 3 1	9.76E-13	3 3 1	9.76E-14
0 3 2	3.25E-12	1 3 2	3.25E-13	2 3 2	3.25E-14	3 3 2	3.25E-15
0 3 3	1.08E-13	1 3 3	1.08E-14	2 3 3	1.08E-15	3 3 3	1.10E-16

For the arrival rate λ from 0.01 to 0.05 and service rate from 0.1to 0.5 the average number of customers and the average waiting time of couples are calculated in Table 7 and Table 8. From Figure 6 and Figure 7 it is clear that as the arrival rate increases the number of customers and the waiting time increases.

Table 7

λ/μ_1	0.1	0.2	0.3	0.4	0.5
0.01	0.1470	0.0885	0.0704	0.0615	0.0563
0.02	0.3243	0.1854	0.1457	0.1269	0.1160
0.03	0.5440	0.2919	0.2265	0.1965	0.1792
0.04	0.8263	0.4096	0.3134	0.2707	0.2466
0.05	1.2072	0.5405	0.4072	0.3501	0.3183

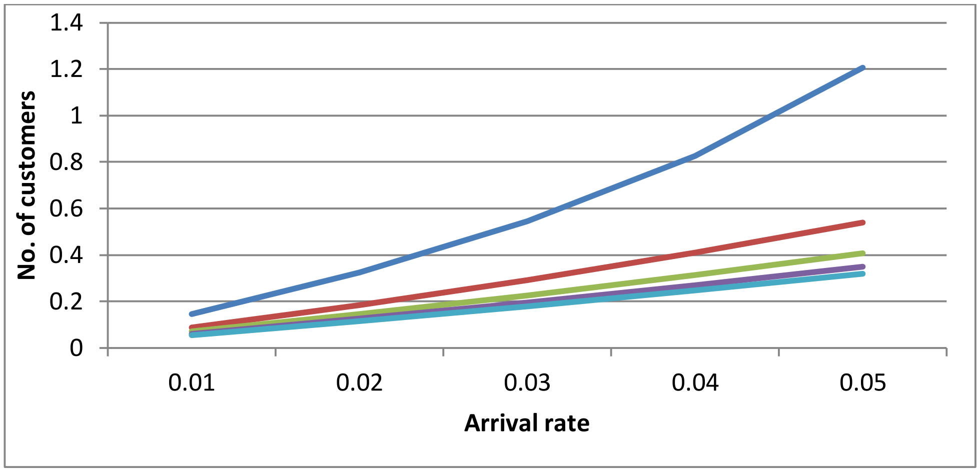

Figure 6

Table 8

λ/μ_1	0.1	0.2	0.3	0.4	0.5
0.01	14.7	8.85	7.04	6.15	5.63
0.02	16.215	9.27	7.285	6.345	5.8
0.03	18.1333	9.73	7.55	6.55	5.9733
0.04	20.6575	10.24	7.835	6.7675	6.165
0.05	24.144	10.81	8.144	7.002	6.366

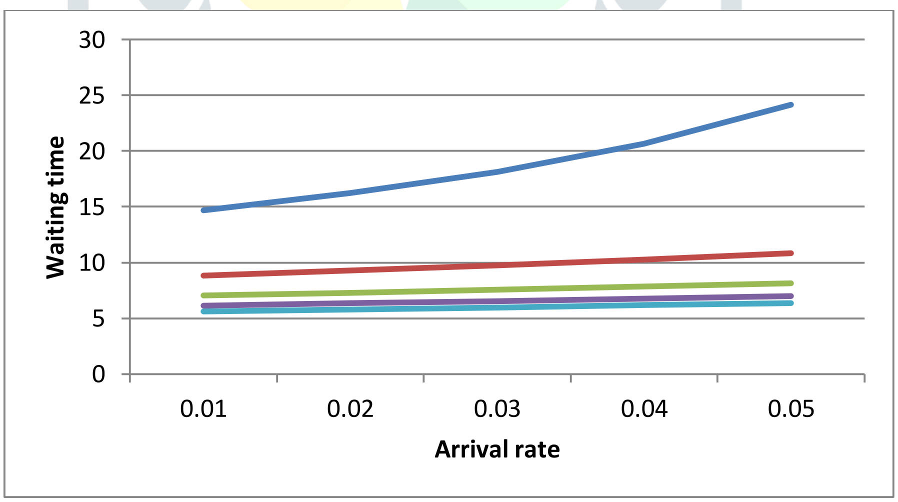

Figure 7

For the service rate from 0.1 to 0.5 and the arrival rate λ from 0.01 to 0.05 the average number of customers and the average waiting time of couples are calculated in Table 9 and Table 10. From Figure 8 and Figure 9 it is clear that as the service rate increases the number of customers and the waiting time decreases.

Table 9

μ_1/λ	0.01	0.02	0.03	0.04	0.05
0.1	0.1470	0.3243	0.5440	0.8263	1.2072
0.2	0.0885	0.1854	0.2919	0.4096	0.5405
0.3	0.0704	0.1457	0.2265	0.3134	0.4072
0.4	0.0615	0.1269	0.1965	0.2707	0.3501
0.5	0.0563	0.1160	0.1792	0.2466	0.3183

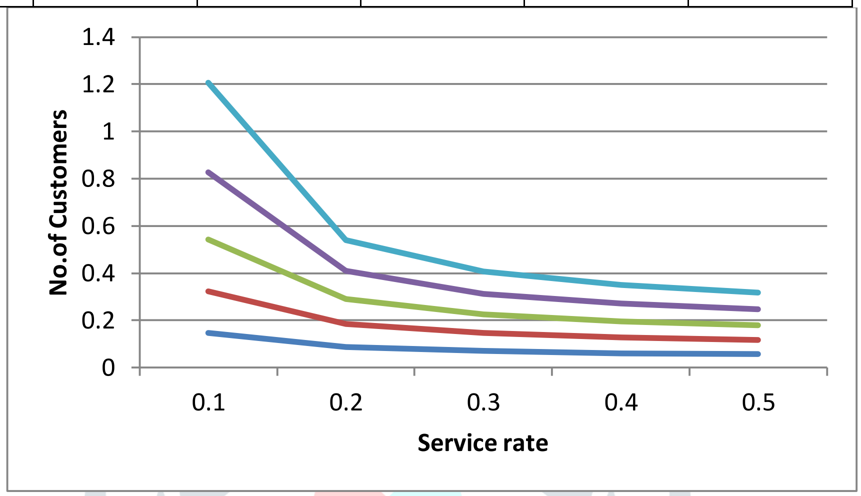

Figure 8

Table 10

μ_1/λ	0.01	0.02	0.03	0.04	0.05
0.1	14.7	16.215	18.1333	20.6575	24.144
0.2	8.85	9.27	9.73	10.24	10.81
0.3	7.04	7.285	7.55	7.835	8.144
0.4	6.15	6.345	6.55	6.7675	7.002
0.5	5.63	5.8	5.9733	6.165	6.366

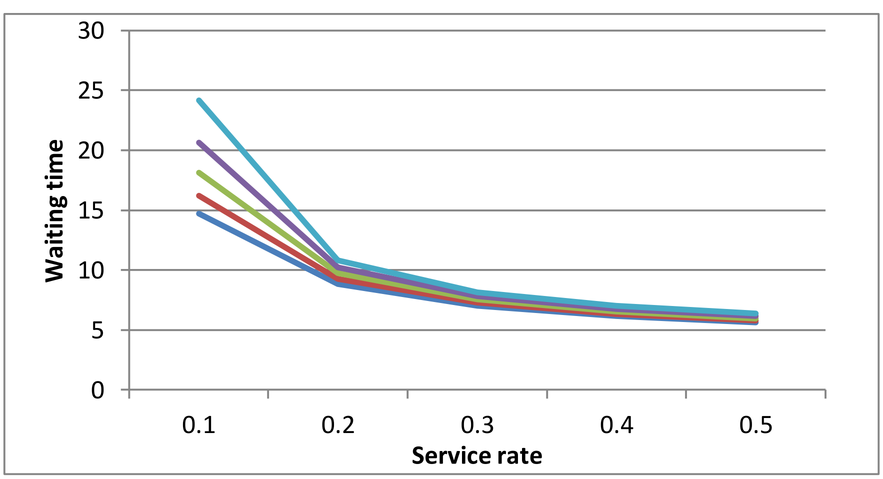

Figure 9

V CONCLUSION

In this paper we studied the queueing model in a healthcare system (hospitals).The steady state probabilities are derived for various numbers of customers (couples) in the system. The numerical examples are also given to test the feasibility. The numerical examples shows that as the arrival rate increases the number of customers and the waiting time increases and as the service rate increases the number of customers and the waiting time decreases. It shows the feasibility of the model.

REFERENCES

1. Avishai Mandelbaum, Petar Mom cilovi´C, Nikolaos Trichakis Sarah Kadish, Ryan Leib, Craig A. Bunnell, "Data-driven appointment-scheduling under uncertainty: The case of an infusion unit in a cancer center", submitted to *Management science* (2018).
2. Bose S.K., "An introduction to queueing Systems", Kluwer academic / Plenum publishers, New York, (2002).
3. Erlang A.K., "The theory of probabilities and telephone conversations", *Nyt Jindsskriff Math*, B 20, 33-39, (1909).
4. Gross D. and Harris C., "Fundamentals of queueing theory", John Wiley& Sons, 1998.
5. Jackson J.R., "Job shop-like queueing systems", *Management Science* 10 131-142 (1963).
6. Jackson J.R., "Networks of waiting lines", *Operation Research* 5(4) 518-521 1957.
7. Jackson R.P., Welch J.D., Fry J., "Appointment systems in hospitals and general practice", *J. Oper. Res. Soc.* 15(3) 219-237 (1964).
8. Mor Armony, Shlomo Israelit, Avishai Mandelbaum, Yariv N. Marmor, Yulia Tseytlin, Galit B. Yom-Tov, "On Patient Flow in Hospitals: A Data-Based Queueing-Science Perspective. An Extended Version", *Stochastic systems*, Vol.5, 146-194, (2015).
9. Ravikant Patel, Hinaben R Patel, "A study on waiting time and out-patient satisfaction at Gujarat medical education research society hospital, Valsad, Gujarat, India" *International Journal of community medicine and public health*, 4(3):857-863 (2017).
10. Sreekala M.S. and Manoharan M., "A queueing network model with feedback and its application in healthcare", *International Journal of new technologies in science & Engineering* Vol.3 Issue 12, 78-89 (2016).
11. Stefan Creemers and Marc R. Lambrecht, "Modeling a Healthcare System as a Queueing Network: The Case of a Belgian Hospital", *SSRN Electronic Journal*, (2007).