Generalized $*$-Lie Ideal of $*$-prime ring

A . R. Gotmare

Department of Mathematics,

GDM Arts KRN Commerce and MD Science College Jamner Dist Jalgaon.

Abstract : Let R be a $*$ - prime ring with characteristic not 2 , $\sigma,\tau : R \to R$ be two automorphisms , U be non zero $*$_ (σ,τ)_Lie ideal of R such that τ commutes with $*$, and $a,b \in R$. (i) If $a \in S_*(R)$ and $[U,a]=0$, then $a \in Z(R)$ or $U \subset Z(R)$ (ii) If $a \in S_*(R)$ and $[U,a] \subset C_{\sigma,\tau}$, then $a \in Z(R)$ or $U \subset Z(R)$.

1 .Introduction : Let R be an associative ring with centre $Z(R)$. Recall that a ring R is prime if $aRb=0 \Rightarrow a=0$ or $b=0$. An involution $*$ of a ring R is an additive mapping satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x,y \in R$ .A ring equipped with involution $*$ is said to be $*$ prime if $aRb = a^*Rb = 0$ or $aRb = aRb^* = 0 \Rightarrow a=0$ or $b=0$. R is said to be 2 -torsion free if whenever $2x=0$ with $x \in R$ then $x=0$. $S_*(R)$ Will denote the set of all symmetric and skew symmetric elements of R , ie $S_*(R)=\{x \in R / x^* = \pm x\}$ .An ideal I of R is said to be $*$ ideal if I is invariant under $*$ n , ie $I^* = I$. As usual the commentator $xy-yx$ is denoted by $[x,y]$. An additive mapping $h: R \to R$ is called a derivation if $h(xy)=h(x)y+xh(y)$ holds for $x,y \in R$ .For a fixed $a \in R$, the mapping $I_a : R \to R$ is give by $I_a(x)=[a,x]$ for $x \in R$ is a derivation , which is said to be inner derivation determined by a .Let σ , τ be two mappings on R .The set $C_{\sigma,\tau}=\{c \in R / c\sigma(r)=\tau(r)c\}$ for all $r \in R$ is called the (σ,τ) -centre of R .In particular , $C_{1,1}=Z(R)$ is the centre of R, where $1: R \to R$ is an identity map.. As usual, the (σ,τ) - commentator $x\sigma(y)-\tau(y)x$ will be denoted by $[x,y]_{\sigma,\tau}$ .An additive mapping $d: R \to R$ is called as (σ,τ) - derivation if $d(xy)=d(x)\sigma(y)+\tau(x)d(y)$ holds for all $x,y \in R$. For a fixed $a \in R$, the mapping $d_a : R \to R$ is given by $d_a(x)=[a,x]_{\sigma,\tau}$ for all $x \in R$ is called a (σ,τ) - inner derivation determined by a . The definition of a (σ,τ) - Lie ideal was given in $[4]$. as follows. Let U be an additive subgroup of R . Then (i) U is a (σ,τ) - right Lie ideal of R if $[U,R]_{\sigma,\tau} \subset U$.(ii) U is a (σ,τ)- Left Lie ideal of R if $[R,U]_{\sigma,\tau} \subset U$.(iii) If U is both (σ,τ) -right Lie and Left Lie ideal of R then U is a (σ,τ) - Lie ideal of R .

A (σ,τ) -Lie ideal of R is said to be a $*-(\sigma,\tau)$ - Lie ideal if U is invariant under $*$. i.e. $U^* = U$. Every $*-$ Lie ideal of R is a $*-$ $(1,1)$ - Lie ideal of R where $1: R \to R$ is identity map but every $*-$ (σ,τ) -Lie ideal of R is in general not $*-$ Lie ideal of R .

. Throughout this paper , R will be a $*-$ prime ring , $Z(R)$ will be centre of R , $\sigma,\tau : R \to R$ will be two auto orphisms , $C_{\sigma,\tau}$ will be the (σ,τ) - centre of R , and $S_*(R)$ will be the set of all symmetric and skew symmetric elements of R .

2 Results : For the proof of our results we need the following lemmas.

Lemma 2.1 *Let* R *be a ring and* U *be a non zero* (σ,τ) *- left Lie ideal of* R *and*

$T=\{c\in R/[R,c]_{\sigma,\tau}\subset U\}$ *Then the following holds.*

(i) T *is a subring of* R

(ii) *If* U *is a* (σ,τ) *- right Lie ideal of* R then T is the largest Lie R such $[R,T]\subset U$ and $U\subset T$.

Lemma 2.2 Let R be a ring and U be a non zero (σ,τ) - left Lie ideal of R. Then $R[T(U),\sigma(T(U))]\subset T(U)$ and $[T(U),\tau(T(U))]R\subset T(U)$.

Lemma 2.3 Let R *be a* σ *prime ring with characteristics not* 2 *,* d *be derivation of* R *satisfying* $d\sigma=\pm\sigma d$ *, an* I *be a non zero* σ *ideal of* R *. If* $d^2(I)=0$ *then* $d=0$.

Lemma 2.4 *Let* I *be a non zero* σ *- ideal of a* σ *- prime ring* R *. If* $a,b\in R$ *such that* $aIb=aI\sigma(b)=0$ *then* $a=0$ *or* $b=0$

Lemma 2.5 *Let* I *be non zero* σ *ideal of a* σ *- prime ring* R *and* $a\in R$ *. If* $Ia=0$ *or* $aI=0$ *then* $a=0$.

Lemma 2.6 *Let* R *be a* σ *- prime ring with characteristics not* 2 *,* I *be a non zero* σ *ideal of* R *, and* d *be a non zero* (α,β) *- derivation of* R *such that* β *commutes with* σ *. If* $d(I)\subset C_{\alpha,\beta}$ *then* R *is commutative.*

Lemma 2.7 *Let* R *be a* σ *- prime ring with characteristics not* 2 *,* I *be a non zero* σ *- ideal of* R *,* d *be a* (α,β) *- derivation of* R *such that* β *commutes with* σ *, and* h *a derivation of* R *satisfying* $h\sigma=\pm\sigma h$ *. If* $dh(I)=0$ *and* $h(I)\subset I$ *then* $d=0$ *or* $h=0$

Lemma 2.8 *Let* U *be a non zero* $*-(\sigma,\tau)$ *-left Lie ideal of* R *such that* τ *commutes with* $*$ *. If* $U\subset C_{\sigma,\tau}$ *then* $U\subset Z(R)$.

Lemma 2.9 *Let* U *be a non zero* $*-(\sigma,\tau)$ *-left Lie ideal of* R *such that* τ *commutes with and* $a\in R$ *. If* $Ua=0$ *then* $a=0$ *or* $U\subset Z(R)$.

Lemma 2.10 *Let* U *be a non zero* $*-(\sigma,\tau)$ *- left Lie ideal of* R *such that* τ *commutes with* $*$*. If* $a\in R$ *and* $[U,a]=0$ *then* $[\sigma(U),a]=0$.

Theorem 2.11 *Let* R *be a* $*$ *- prime ring with characteristics not* 2*, and let* U *be a non zero* $*-(\sigma,\tau)$ *-Lie ideal of* R *such that* τ *commutes with* $*$ *. If* $a\in S_*(R)$ *and* $[U,a]$ *=0, then* $a\in Z(R)$ *or* $U\subset Z(R)$ *.*

Proof Let

$$T(U)=\{c\in R/[R,c]_{\sigma,\tau}\subset U\}.$$

Since U is a (σ,τ)- Lie ideal of R, from Lemma 2.1, $T(U)$ is a Lie ideal of R such that $U\subset T(U)$.

Hence, it follows that $[R,U]\subset[R,T(U)]\subset T(U)$.

From the definition of $T(U)$, we have

$$[R,[R,U]]_{\sigma,\tau}\subset[R,T(U)]_{\sigma,\tau}\subset U$$

Rom the hypothesis, we have $\left[R,[R,U]_{\sigma,\tau},a\right]=0$. Foe any $r,s\in R$ and $u\in U$, it holds that

$$\left[r,[s,u]_{\sigma,\tau},a\right]=0 \qquad (i)$$

Replacing r by ra in (i) and expanding by using (i) we obtained $r[[a,\sigma([s,u]),a]]+[r,a][a,\sigma(s,u)]=0$ $\forall r,s\in R$ and $u\in U$. (ii)

In (ii) taking rm instead of r where $m\in R$ and expanding by using (ii) we show that

$$[r,a]R[a,\sigma(s,u)]=0\ \forall r,s\in R \text{ and} \forall r,s\in R\ \forall r,s\in R.$$

Replacing r by r^* and using $a\in S_*(R)$ in the last equation, we have

$$[r,a]^* R[a,\sigma[s,u]]=0$$

By the $*$ - primness of R , we get

$$a\in Z(R) \text{ Or } [a,\sigma(s,u)]=0$$

That is $[\sigma(s,u),a]=0$

Since σ is an auto orphism , it implies

$$[[R,\sigma(u)],a]=0$$

From lemma 2.10 , we have $[\sigma(U,a)]=0.$ It holds that

$$[[r,a],\sigma(u)]=0 \qquad (iii)$$

From (iii) ,it implies

$$\left(I_{\sigma(u)}I_a\right)(R)=0$$

Since $a\in S_*(R)$, we know that $I*_a=*I_{a.}$ Hence according to lemma 2.7, we have

$$a\in Z(R) \text{ or } \sigma(u)\in Z(R) \text{ which implies that } a\in Z(R) \text{ or } U\subset R.$$

Theorem 2.12 *Let R be a $*$ prime ring with characteristics not 2 , and let U be a non zero $*-(\sigma,\tau)$ -Lie ideal of R such that τ commutes with $*$, If $a \in S_*(R)$ and $[U,a] \subset C_{\sigma,\tau} = 0$ then $a \in Z(R)$ or $U \subset Z(R)$.*

References

$[i]$ Aydm N. Notes on Generalized Lie ideals

$[ii]$ Bergen J Lie ideals and derivations of prime rings .

$[iii]$ Kayo K. (σ,τ) - Lie ideals in a prime rings .

$[iv]$ Turkmen S, Aydm N . Some results on σ -prime rings .