COEFFICIENT BOUNDS FOR NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS

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Abstract :

In the present investigation, we consider two new subclasses $\mathcal{N}_{\Sigma}^{\mu}(\alpha,\lambda)$ and $\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda)$ of bi-univalent functions defined in the open unit disk $\mathcal{U}=\{z:|z|<1\}$. Besides, we find upper bounds for the second and third coefficients for functions in these new subclasses.

Keywords:

Bi-univalent functions, Starlike functions with respect to symmetric points, Convex functions with respect to symmetric points, Close-to-convex functions, Quasi- convex functions.

Introduction:

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

Which are analytic in the open unit disk $\mathcal{U}=\{z:|z|<1\}$. Further, by S we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

Ding et al. [8] introduced the following class $Q_\lambda(\beta)$ of analytic functions defined as follows:

$$Q_\lambda(\beta)=\left\{f\in\mathcal{A}: Re\left((1-\lambda)\frac{f(z)}{z}+\lambda f'(z)\right)>\beta,\beta<1,\lambda\geq 0\right\}.$$

It is easy to see that $Q_{\lambda_1}(\beta)\subset Q_{\lambda_2}(\beta)$ for $\lambda_1>\lambda_2\geq 0$.Thus, for $\lambda\geq 1, 0\leq\beta<1, Q_\lambda(\beta)\subset Q_1(\beta)=\{f\in\mathcal{A}: Ref'(z)>\beta, 0\leq\beta<1\}$ and hence $Q_\lambda(\beta)$ is univalent class.

It is well known that every function $f\in S$ has an inverse f^{-1}, defined by

$$f^{-1}\big(f(z)\big)=z \quad (z\in\mathcal{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f) \geq \frac{1}{4}\right)$$

Where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots$$

Brannan and Taha introduced certain subclasses of the bi-univalent function class $\sum$ similar to the familiar subclasses $S^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α ($0 \leq \alpha < 1$),respectively. Thus following Brannan and Taha, function $f \in \mathcal{A}$ is in the class $S_\Sigma^*(\alpha)$ of strongly bi-starlike function of orderα $(0 < \alpha \leq 1)$if each of the following conditions are satisfied:

$$f \in \Sigma \text{and} \left|arg\left(\frac{zf\prime(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ z \in \mathcal{U})$$

and

$$\left|arg\left(\frac{zg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2}(0 < \alpha \leq 1, \qquad w \in \mathcal{U}),$$

Where g is the extension of $f^{-1}$to $\mathcal{U}$.

The classes $S_\Sigma^*(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi- starlike functions of order α and bi-convex functions of order α, corresponding (respectively)to the function classes $S^*(\alpha)$ and $\mathcal{K}(\alpha)$, were also introduced analogously. For each of the function classes $S_\Sigma^*(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$(for details, see[4,25]).

The object of the present paper is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ employing the techniques used earlier by Srivastava et al.

In order to derive our main results, we need to following lemma.

Lemma:

If $p\epsilon\mathcal{P}$, then $|C_k \leq 2|$ for each k, where P is the family of all functions p analytic in u for which $\mathcal{R}_p(z) > 0, p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ for $z\epsilon U$.

Definition

A function $f(z)$ is said to be in the class $\mathcal{N}_\Sigma^\mu(\alpha, \lambda)$ if the following conditions are satisfied.

$$f \in \Sigma \text{and} \left|\arg((1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1})\right| < \frac{\alpha\pi}{2}, \quad (0 < \alpha \leq 1, \mu \geq 0, z \in \mathcal{U})$$

and

$$\left|\arg((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1})\right|<\frac{\alpha\pi}{2},\quad (0<\alpha\leq 1,\mu\geq 0,w\in\mathcal{U})$$

Where the function g is given by

$$g(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots$$

Note that for $\lambda=\mu=1$, the class $\mathcal{N}_{\Sigma}^{1}(\alpha,1)$ introduced andstudied by srivastava et al and for $\mu=1$, the class $\mathcal{N}_{\Sigma}^{1}(\alpha,\lambda)$ introduced and studied by Frasin and Aouf

Theorem Let $f(z)$ given by (1.1.1) be in the class $\mathcal{N}_{\Sigma}^{\mu}(\alpha,\lambda)$, $o<\alpha\leq 1,\lambda\geq 1$and $\mu\geq 0$. Then

$$|a_2|\leq\frac{2\alpha}{\sqrt{(\lambda+\mu)^2+\alpha(\mu+2\lambda-\lambda^{2)}}}$$

and

$$|a_3|\leq\frac{4\alpha^2}{(\lambda+\mu)^2}+\frac{2\alpha}{2\lambda+\mu}$$

Proof:

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}=[p(z)]^{\alpha}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}=[q(w)]^{\alpha}$$

where

$$p(z)=1+p_1z+p_2z^2+\cdots$$

and

$q(w)=1+q_1w+q_2w^2+\cdots$in $\mathcal{P}$.

Now,equality the coefficients we have,

$$\Rightarrow(1-\lambda)\left(\frac{z+a_2z^2+a_3z^3+\cdots}{z}\right)^{\mu}+\lambda(1+2a_2z+3a_3z^3+\cdots)$$

$$\left(\frac{z+a_2z^2+a_3z^3+\cdots}{z}\right)^{\mu-1} = [1+p_1z+p_2z^2+\cdots]^{\alpha}$$

$$(1-\lambda)(1+a_2z+a_3z^2+\cdots)^{\mu}+\lambda(1+2a_2z+3a_3z^2+\cdots)(1+a_2z+a_3z^2+\cdots)^{\mu-1}$$

$$=[1+p_1z+p_2z^2+\cdots]^{\alpha}$$

$$(1-\lambda)\left[1+\frac{\mu}{1!}(a_2z+a_3z^2+\cdots)+\frac{\mu(\mu-1)}{2!}(a_2z+a_3z^2+\cdots)^2\right]$$

$$+\lambda(1+2a_2z+3a_3z^2+\cdots)\left[1+\frac{\mu-1}{1!}(a_2z+a_3z^2+\cdots)+\frac{\mu-1(\mu-2)}{2!}(a_2z+a_3z^2+\cdots)^2\right]$$

$$=1+\frac{\alpha}{1!}(p_1z+p_2z^2+\cdots)+\frac{\alpha(\alpha-1)}{2!}(p_1z+p_2z^2+\cdots)^2+\cdots$$

$$(1-\lambda)+(1-\lambda)\mu(a_2z)+(1-\lambda)\mu(a_3z^2)+\cdots+\frac{\mu(\mu-1)}{2}a_2^2z^2+\frac{\mu(\mu-1)}{2}a_3^2z^4$$

$$+\mu(\mu-1)a_2a_3zz^2+\cdots+\lambda+2\lambda a_2z+3\lambda a_3z^2+(\lambda+2\lambda a_2z+3\lambda a_3z^2)\frac{\mu-1}{1}(a_2z+a_3z^2+\cdots)$$

$$+(\lambda+2\lambda a_2z+3\lambda a_3z^2)\frac{(\mu-1)(\mu-2)}{2!}(a_2^2z^2+a_3^2z^4+2a_2a_3z^3$$

$$=1+\alpha p_1z+\alpha p_2z^2+\frac{\alpha(\alpha-1)}{2}({p_1}^2z^2+{p_2}^2z^4+2p_1p_2z^3+\cdots)$$

$$\Rightarrow \qquad (1-\lambda)\mu(a_2z)+2\lambda a_2z+\lambda(\mu-1)a_2z=\alpha p_1z$$

$$\mu a_2z-\lambda\mu a_2z+2\lambda a_2z+\lambda\mu a_2z-\lambda a_2z=\alpha p_1z$$

$$\mu a_2z+\lambda a_2z=\alpha p_1z$$

$\Rightarrow(\lambda+\mu)a_2=\alpha p_1,$

$$(2\lambda+\mu)a_3+(\mu-1)\left(\lambda+\frac{\mu}{2}\right)a_2^2=\alpha p_2+\frac{\alpha(\alpha-1)}{2}{p_1}^2,$$

$$(3.4)\Rightarrow(1-\lambda)\left(\frac{w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots}{w}\right)^{\mu}$$

$$+\lambda(1-2a_2w+3(2a_2^2-a_3)w^2 -\cdots)\left(\frac{w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots}{w}\right)^{\mu-1}$$

$$=[1+q_1w+q_2w^2+\cdots]^{\alpha}$$

$$\Rightarrow(1-\lambda)[1-a_2w+(2a_2^2-a_3)w^2+\cdots]^{\mu}$$

$$+(\lambda-2\lambda a_2w+3\lambda(2a_2^2-a_3)w^2-\cdots)[1-a_2w+(2a_2^2-a_3)w^2+\cdots]^{\mu-1}$$

$$=1+\frac{\alpha}{1!}(q_1w+q_2w^2+\cdots)+\frac{\alpha(\alpha-1)}{2!}(q_1w+q_2w^2+\cdots)^2+\cdots$$

$\Rightarrow(1-\lambda)-\mu(1-\lambda)a_2w+\mu(1-\lambda)(2a_2^2-a_3)w^2-\frac{\mu(\mu-1)}{2}(1-\lambda)(-a_2w+(2a_2^2-a_3)w^2-\cdots)(\lambda-2\lambda a_2w+3\lambda(2a_2^2-a_3)w^2-\cdots)(1+(\mu-1)(-a_2w+(2a_2^2-a_3)w^2-\cdots)+\frac{(\mu-1)(\mu-2)}{2}(2a_2^2-a_3)w^2-\cdots)$

$$=1+\alpha q_1w+\alpha q_2w^2+\cdots+\frac{\alpha(\alpha-1)}{2}q_1^2w^2+\frac{\alpha(\alpha-1)}{2!}q_2^2w^4+\frac{\alpha(\alpha-1)}{2!}2q_1q_2w^3+\cdots$$

$\Rightarrow-(\lambda+\mu)a_2=\alpha q_1$

and

$$-(2\lambda+\mu)a_3+(3+\mu)\left(\lambda+\frac{\mu}{2}\right)a_2^2=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2$$

From we obtain

$$p_1=-q_1$$

and

$$2(\lambda+\mu)^2a_2^2=\alpha^2(p_1^2+q_1^2)$$

Now, from

$\Rightarrow(2\lambda+\mu)a_3+(\mu-1)\left(\lambda+\frac{\mu}{2}\right)a_2^2-(2\lambda+\mu)a_3+(3+\mu)\left(\lambda+\frac{\mu}{2}\right)a_2^2$

$$=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2+\frac{\alpha(\alpha-1)}{2}q_1^2$$

$\Rightarrow\left(\lambda+\frac{\mu}{2}\right)a_2^2[(\mu-1)+(3+\mu)]=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)$

$$\frac{(2\lambda+\mu)}{2}[\mu-1+3+\mu]a_2^2=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)$$

$$\frac{(2\lambda+\mu)}{2}[2\mu+2]a_2^2=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)$$

$$(2\lambda+\mu)(\mu+1)a_2^2=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)$$

Now applying the value of $(p_1^2+q_1^2)$ from, we get

$$(2\lambda+\mu)(\mu+1)a_2^2=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}\frac{2(\lambda+\mu)^2}{\alpha^2}a_2^2$$

$$=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{\alpha}(\lambda+\mu)^2a_2^2$$

Therefore, we have

$$\left[(2\lambda+\mu)(\mu+1)-\frac{\alpha(\alpha-1)}{\alpha}(\lambda+\mu)^2\right]a_2^2=\alpha(p_2+q_2)$$

$$\frac{1}{\alpha}[\alpha(2\lambda\mu+\mu^2+2\lambda+\mu)-(\alpha-1)(\lambda^2+\mu^2+2\lambda\mu)]a_2^2=\alpha(p_2+q_2)$$

$$\frac{1}{\alpha}[2\alpha\lambda\mu+\alpha\mu^2+2\alpha\lambda+\alpha\mu-\alpha\lambda^2+\alpha\mu+2\alpha\lambda\mu+\lambda^2+\mu^2+2\lambda\mu]=\alpha(p_2+q_2)$$

$$\frac{1}{\alpha}[\alpha(\mu^2+2\lambda-\lambda^2)+(\lambda+\mu)^2]=\alpha(p_2+q_2)$$

$\Rightarrow[\,\alpha(\mu^2+2\lambda-\lambda^2)+(\lambda+\mu)^2]a_2^2=\alpha^2(p_2+q_2)$

$\Rightarrow a_2^2=\frac{\alpha^2(p_2+q_2)}{\alpha(\mu^2+2\lambda-\lambda^2)+(\lambda+\mu)^2}$

Applying lemma we define

$$|a_2|\le\sqrt{\frac{\alpha^2(p_2+q_2)}{\alpha(\mu^2+2\lambda-\lambda^2)+(\lambda+\mu)^2}}$$

$$|a_2|\le\frac{2\alpha}{\sqrt{\alpha(\mu^2+2\lambda-\lambda^2)+(\lambda+\mu)^2}}$$

Which gives us desired estimate on $|a_2|$ as asserted,

Next, in order to find the bound on $|a_3|$, by subtracting we get

$$(2\lambda+\mu)a_3+(\mu-1)\left(\lambda+\frac{\mu}{2}\right)a_2^2+(2\lambda+\mu)a_3-(3+\mu)\left(\lambda+\frac{\mu}{2}\right)a_2^2$$

$$=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2-(\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2)$$

$$2(2\lambda+\mu)a_3+[(\mu-1)\frac{(2\lambda+\mu)}{2}-(3+\mu)\left(\lambda+\frac{\mu}{2}\right)]a_2^2$$

$$=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2-\left(\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2\right)$$

$$2(2\lambda+\mu)a_3+\frac{(2\lambda+\mu)}{2}[\mu-1-3-\mu]\,a_2^2$$

$$=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2-\left(\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2\right)$$

$$2(2\lambda+\mu)a_3-2(2\lambda+\mu)a_2^2$$

$$=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2-\left(\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2\right)$$

It follows

$$2(2\lambda+\mu)a_3-2(2\lambda+\mu)\frac{\alpha^2(p_1^2+q_1^2)}{2(\lambda+\mu)^2}=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2)$$

$$2(2\lambda+\mu)\left[a_3-\frac{\alpha^2(p_1^2+q_1^2)}{2(\lambda+\mu)^2}\right]=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}((-q_1^2)-q_1^2)$$

$\Rightarrow 2(2\lambda+\mu)\left[a_3-\frac{\alpha^2(p_1^2+q_1^2)}{2(\lambda+\mu)^2}\right]=\alpha(p_2-q_2)+0$

$\Rightarrow 2(2\lambda+\mu)\left[a_3-\frac{\alpha^2(p_1^2+q_1^2)}{2(\lambda+\mu)^2}\right]=\alpha(p_2-q_2)$

$\Rightarrow a_3=\frac{\alpha^2(p_1^2+q_1^2)}{2(\lambda+\mu)^2}+\frac{\alpha(p_2-q_2)}{2(2\lambda+\mu)}$

Applying lemma we readily get

$$|a_3|\le\frac{4\alpha^2}{(\lambda+\mu)^2}+\frac{2\alpha}{(2\lambda+\mu)}$$

Hence completes the proof of theorem

Corollary:

Let $f(z)$ given by (1.1.1) be in the class $\mathcal{N}_{\Sigma}^{\mu}(\alpha,\lambda)$, $0\le\alpha\le1$ and $\lambda\ge1$. Then

$$|a_2|\le\frac{2\alpha}{\sqrt{\alpha(1+2\lambda-\lambda^2)+(\lambda+1)^2}}$$

$$\text{and}|a_3| \le \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{(2\lambda+1)}$$

Corollary :

Let $f(z)$ given by (1.1.1) be in the class $\mathcal{N}_{\Sigma}^{\mu}(\alpha,\lambda)$, $0 \le \alpha \le 1$. Then

$$|a_2| \le \alpha\sqrt{\frac{2}{\alpha+2}}$$

$$\text{and}|a_3| \le \frac{\alpha(3\alpha+2)}{3}$$

Corollary :

Let $f(z)$ given by (1.1.1) be in the class $S_{\Sigma}^{*}[\alpha]$, $0 \le \alpha \le 1$. Then

$|a_2| \le \frac{2\alpha}{\sqrt{1+\alpha}}$and $|a_3| \le \alpha(4\alpha+1)$.

Conclusion:

In our present study we considered two new subclasses $\mathcal{N}_{\Sigma}^{\mu}(\alpha,\lambda)$ and $\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda)$ of bi-univalent functions in the open unit disk$\mathcal{U} = \{z: |z| < 1\}$.We have investigated estimates on the first two Taylor-Maclaurin coefficient $|a_2|\&|a_3|$ for functions belonging to this classes. we have shown already that the results and corollaries presented in this paper would generalize and improve some recent works of Srivastava et al.[12], Frasin and Aouf [7] and Caglar et al.[5].

References

[1] D.A. Brannan, J. Clunie, W.E. Kirwan, Coefficient estimates for a class of starlike functions, Canad. J. Math. 22 (1970) 476–485.
[2] D.A. Brannan, J.G. Clunie (Eds.), Aspects of Contemporary Complex Analysis (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1 20, 1979), Academic Press, New York and London, 1980.
[3] D.A. Brannan, T.S. Taha, On some classes of bi-univalent functions, in: S.M. Mazhar, A. Hamoui, N.S. Faour (Eds.), Math.Anal.and Appl., Kuwait; February 18–21, 1985, in: KFAS Proceedings Series, vol. 3, Pergamon Press, Elsevier Science Limited, Oxford, 1988, pp. 53–60. see also Studia Univ. Babe¸s-Bolyai Math. 31 (2) (1986) 70–77.
[4] B.A. Frasin, M.K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett. 24 (2011) 1569–1573.
[5] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. 18 (1967) 63-68.
[6] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in |z| < 1, Arch. Rational Mech. Anal. 32 (1969) 100-112.
[7] Ch. Pommerenke, Univalent Functions, Vandenhoeck and Rupercht, G¨ottingen, 1975.
[8] H.M. Srivastava, A.K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. 23 (2010) 1188–1192.
[9] T.S. Taha, Topics in univalent function theory, Ph.D. Thesis, University of London, 1981.
[10] Y. Zhu, Some starlikeness criterions for analytic functions, J. Math. Anal. Appl. 335 (2007)