# GENERALIZATION OF COMMON FIXED POINT THEOREM FOR A BANACH OPERATOR PAIR OF MAPPINGS IN A CONE METRIC SPACE

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***Abstract:*** The purpose of this paper, we establish and generalize common fixed point theorems for a Banach operator pairs of mappings satisfying contraction condition in the setting of complete cone metric spaces. Our results generalize and extend some well known results in the literature of [41].

**Keywords:** Cone metric space, complete cone metric space, fixed point, common fixed point, Banach operator pair.

## I. Introduction

The first fundamental result of fixed point theory is Banach contraction principle, which introduced in 1922 by Banach [1] as the following theorem:-

**Theorem1.** Let $(X,d)$be a complete metric space and let $T: X \to X$be Banach contraction mapping, if there exist a constant $\alpha \in [0,1)$ such that

$$d(Tx,Ty) \leq \alpha d(x,y) \text{ , for all } x,y \in X.$$

Then T has a unique fixed point. It is one of the famous and traditional theorems in modern mathematics which is widely applied in many other branches of science and applied science.

In 1968 and 1969, Kannan [2, 3] introduced the concept of Kannan mappings as follows:-**Theorem2.** Let $(X,d)$be a complete metric space and let $T: X \to X$be Kannan contraction mapping, if there exist a constant $b \in [0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq a[d(x,Tx) + d(y,Ty)] \text{ , for all } x,y \in X.$$

Then T has a unique fixed point.

Chatterjea [4], introduced the concept of chatterjea contraction mapping in 1972, as follows:-

**Theorem3.** let $T: X \to X$be Chatterjea contraction mapping on complete metric space $(X,d)$ and if there exist a constant $c \in [0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)] \text{ , for all } x,y \in X.$$

Then T has a unique fixed point.

In 1972, Zamfirescu [5], Introduced the concept of Zamfirescu mapping as follows:

**Theorem4.** Let $(X,d)$be a complete metric space and let $T: X \to X$be a Zamfirescu contraction mapping, if there exist a constant $\alpha \in [0,1), \beta \in \left[0,\frac{1}{2}\right)$ and$\gamma \in [0,\frac{1}{2})$ such that at least one of the following conditions is true.

$$(z_1)\ d(Tx,Ty) \leq \alpha d(x,y) \text{ ,}$$

$$(z_2)\ d(Tx,Ty) \leq \beta[d(x,Tx) + d(y,Ty)] \text{ ,}$$

$$(z_3)\ d(Tx,Ty) \leq \gamma[d(x,Ty) + d(y,Tx)] \text{ , for all } x,y \in X.$$

Then $T$ has a unique fixed point. In The same way, this principle have studied and generalized by several authors in various directions in the same literature.

The notion of Cone metric spaces was introduces in 2007 byHuang and Zhang [6] , which is generalization of metric space. He replaced real number system by ordered Banach space and showed some fixed point theorems of different type of contractive mappings on cone metric spaces. Subsequently, many authors generalized and studied fixed and common fixed point results in cone metric spaces for normal and non normal cone see for instance ([7-32]). Afterwards, Subramanyan [33] gave introduced and called Banach operator of type k and obtained the fixed point in complete metric space. Recently, Chen and Li[34] extended the concept of Banach operator of type k to Banach operatoe pair and proved various best approximation results using common fixed point theorems for f- non expansive mappings. Al-thagafi and Shahzad[35] and Hussain [38] generalized the results of Chen and Li[33]. In [36], authors have proved some common fixed point theorems for a Banach pair of mapping satisfying T-Hardy Rogers type contraction condition in cone metric spaces. In sequel, Ozturk and Basarir [40], proved some common fixed point theorems f-contraction mappings in cone metric spaces without the assumption of normality condition of the cone. In 2014, Dubey et al. [39] generalized the results of [36] and proved some common fixed point theorems for generalized T-Hardy Rogers contraction condition in cone metric spaces to the case of Banach operator pair. In sequel, Raghvendra et al. [37] have proved common fixed point theorems for two Banach pairs of mapping which satisfying contraction conditions in cone metric spaces.

The aim of this paper is to prove common fixed point theorems for two Banach pair of mappings which satisfying contraction conditions in cone metric spaces, which is generalization of results of [27,41]by assumption of normality condition of the cone.

## II. PRELIMINARY NOTES

First, we recall some standard notations and definitions which we needed them in the sequel.

**Definition 2.1([6]):** Let $E$ be a real Banach space and $P$ be a subset of $E$and 0denote to the zero element in $E$, then $P$ is called a cone if and only if :

(i) $P$ is a non-empty set closed and $P \neq \{0\}$,

(ii) If $a, b$ are non-negative real numbers and $x, y \in P$,then $ax + by \in P$,

(iii) $x \in P \; and - x \in P \Longrightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone P ⊂ E, we define a partial ordering ≤ on $E$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$.We shall write $x \ll y$ if $y - x \in intP$(where int $P$ denotes the interior of $P$). If $intP \neq \emptyset$, then cone $P$ is solid. The cone $P$ called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \;\; => \parallel x \parallel \leq k \parallel y \parallel.$$

The least positive number k satisfying the above is called the normal constant of $P$.

**Definition: 2.2([6]):** Let $X$be a non-empty set. Suppose $E$ is a real Banach space, $P$ is a cone with $intP \neq \emptyset$ and $\leq$ is a partial ordering with respect to $P$.If the mapping $d: X \times X \longrightarrow E$ satisfies

(i) $0 < d(x, y) for \; all \; x, y \in X \; and \; (x, y) = 0$if and only if $x = y$,

(ii) $d(x, y) = d(y, x) for \; all \; x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y) for \; all \; x, y, z \in X$.

Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space .The concept of cone metric space is more general than that of a metric space.

**Example2.3**: Let $E = R^2, P = \{(x, y) \in E: x, y \geq 0\}, X = R$ and $d: X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$,where $\alpha \geq 0$ is a constant. Then $(X, d)$ is a cone metric space.

**Definition: 2.4([20])**: Let $(X, d)$ be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ be a sequence in$X$. Then,

(1) $\{x_n\}_{n\geq 1}$ Converges to $x$ whenever for every$c \in E$ with $\theta \ll c$, if there is a natural Number $N$ such That $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $lim_{n\to\infty} x_n = x$ $or$ $x_n \to x, (n \to \infty)$

(2) $\{x_n\}_{n\geq 1}$is said to be a Cauchy sequence if for every $c \in E$ with $\theta$<< c,if there is Natural number $N$ such that $d(x_n, x_m) \ll c$ for all $n.m \geq N$.

(3) $(X, d)$ is called a complete cone metric space if every Cauchy sequence in $X$ is Convergent.

**Definition 2.5:** A self mapping $T$ of a metric space $(X, d)$ is a said to be contraction mapping. If there exist a real number $0 \leq k < 1$ such that for all $x, y \in X$.

$$d(Tx, Ty) \leq kd(x, y).$$

The following definition is given by Beiranvand et ai. [16].

**Definition: 2.6([42])**: Let $T$ and $f$ be any two self mapping of a metric space$(X, d)$. The self mapping $f$ of $X$ is said to be $T$-contraction, if there exist a real number $0 \leq k < 1$ such that

$$d(Tfx, Tfy) \leq kd(Tx, Ty) \text{ for all } x, y \in X.$$

If T= I, the identity mapping, then the definition 2.6 reduce to Banach contraction mapping.

**Example 2.7:** Let $X = [0, \infty)$be with the usal metric. Let define two Mappings $T, f: X \to X$as

$$fx = \beta x, \beta > 1$$

$$Tx = \frac{\alpha}{x^2}, \alpha \in R.$$

It is clear that, f is not contraction but f is T- contraction, since

$$d(Tfx, Tfy) \leq \left|\frac{\propto}{\beta^2 x^2} - \frac{\propto}{\beta^2 y^2}\right| = \frac{1}{\beta}|Tx - Ty|.$$

**Definition 2.8 ([42]):** Let$(X, d)$ be a metric space, and let $T: X \to X$ be self mapping in X. Then

i) A mapping $T$ is said to be sequentially convergent if the sequence $\{y_n\}$ in X is convergent whenever $\{Ty_n\}$ is also convergent.

ii) A mapping $T$ is said to be sub sequentially convergent, if $\{y_n\}$ has whenever $\{Ty_n\}$ is Convergent.

**Definition 2.9([33]):** Let $T$ be a self mapping of a normed space X. Then T is called a Banach operator of type k if

$$\|T^2x - Tx\| \leq k\|Tx - x\|, \text{ for some } k \geq 0 \text{ and for all} x \in X.$$

This concept was introduced by Subrahmanyam[33], then Chen and Li[34]extended this as following:

**Definition 2.10([34]):** Let $T$ and $f$ be any two self mapping of a non empty subset $M$ of a normed space $X$. Then$(T, f)$ is a Banach operator pair, if any one of the following conditions is satisfied:

(i). $T(F(f) \subseteq F(f)$ $i.e$ $F(f)$ is T-invariant.

(ii). $fTx = Tx$ for each $x \in F(f)$.

(iii). $fTx = Tfx$ for each $x \in F(f)$.

(iv). $\|Tfx - fx\| \leq k\|fx - x\|$ for some $k \geq 0$.

**Remark 2.11([16]):** If $c \in intP, 0 \leq a_n$ and $a_n \to 0$, then there exist $n_0$ such that $a_n \ll c$ for all $n > n_0$.

## III. Main Results.

In 2018, Petwal and Dimri proved [41] the following theorem:

**Theorem 3.1:** Let $T$ and $f$ be two continuous self-mappings of a complete cone metric space $(X,d)$. Assume that $T$ is an injective mapping and $P$ is a normal cone with normal constant. If the mappings $T$ and $f$ satisfying

$$d(T^p fx,T^p fy)\leq a[d(T^p x,T^p fx)+d(T^p y,T^p fy)] \quad (3.1)$$

for all x, y ∈ X, where $p$ is a positive integer and $a\in(0,1/2)$. Then $f$ has a fixed point in $X$. Moreover, if $(T, f)$ is a Banach pair, then $T$ and $f$ have unique common fixed point in $X$.

**Next we generalize and extend this theorem in cone metric spaces as the theorems 3.2, 3.3, 3.4 &3.5.**

**Theorem 3.2**: Let $(X,d)$be cone metric spaces and let $T,T_1,T_2: X\rightarrow X$ be any three continuous self mappings on $X$. Assume that $T$ is an injective maps and $P$ is a normal cone with normal constant. If the mapping $T,T_1$ and $T_2$ satisfy the condition

$$d(T^pT_1x,T^pT_2y)\leq\lambda[d(T^px,T^pT_1x)+d(T^py,T^pT_2y)]\ldots\ldots\ldots \quad (3.2.1)$$

for all$x,y\in X$ where p is a positive integer and $\propto\in[0,\frac{1}{2})$. Then $T_1$ and $T_2$have an unique common fixed point in $X$. Moreover, if $(T,T_1)$ and $(T,T_2)$ are Banach pair, then $T,T_1$ and $T_2$ have a unique common fixed point in $X$.

Proof: Let $x_0$ be an arbitrary point in $X$. We define the iterative sequence$\{x_{2n}\}$ and $\{x_{2n+1}\}$by

$$x_{2n+1}=T_1x_{2n}={T_1}^{2n}x_0\ldots\ldots\ldots\ldots \quad (3.2.2)$$

and

$$x_{2n+2}=T_2x_{2n+1}=T_2{}^{2n+1}x_0\ldots\ldots\ldots\ldots \quad (3.2.3).$$

Then from (3.1.1) we have

$$\begin{aligned} d(T^px_{2n+1},T^px_{2n}) &= d(T^pT_1\,x_{2n},T^pT_2\,x_{2n-1})\\ &\leq\lambda[d\big(T^px_{2n},T^pT_1x_{2n}\big)+d(T^px_{2n-1},T^pT_2x_{2n-1})]\\ &\leq\lambda[d\big(T^px_{2n},T^px_{2n+1}\big)+d(T^px_{2n-1},T^px_{2n})]\end{aligned}$$

So, $$\begin{aligned} d(T^px_{2n+1},T^px_{2n}) &\leq \frac{\lambda}{1-\lambda}\,d(T^px_{2n-1},T^px_{2n})\\ &\leq Ld(T^px_{2n},T^px_{2n-1})\ldots\ldots\ldots\ldots \quad (3.2.4)\end{aligned}$$

Where $\frac{\lambda}{1-\lambda}=L<1$.

In general, by induction we have

$$d(T^px_{2n+1},T^px_{2n})\leq Ld(T^px_{2n},T^px_{2n-1})\leq\ldots\ldots\leq L^{2n}d(T^px_1,T^px_0),\text{ for } n\geq 0.$$

So, for $m,n\in N$ with $n>m$ we have

$$\begin{aligned} d(T^px_{2n},T^px_{2m}) &\leq d\big(T^px_{2n},T^px_{2n-1}\big)+d\big(T^px_{2n-1},T^px_{2n-2}\big)+\ldots\ldots+d(T^px_{2m+1},T^px_{2m})\\ &\leq(L^{2n-1}+L^{2n-2}+\cdots\ldots\ldots\ldots\ldots\ldots+L^{2n})d(T^px_1,T^px_0),\\ &\leq\frac{L^{2n}}{1-L}\,d(T^px_1,T^px_0),\ \ldots\ldots\ldots\ldots \quad .(3.2.5)\end{aligned}$$

Since $P$ is normal cone with normal constant, so by (3.1.5) we get

$$\|d(T^px_{2n},T^px_{2m})\|\leq\frac{L^{2n}}{1-L}\,\|d(T^px_1,T^px_0)\|\ldots\ldots\ldots \quad (3.2.6)$$

Since $k\in(0,1)\Rightarrow k\rightarrow 0$ $as$ $n\rightarrow\infty$. Therefore $\|d(T^px_{2n},T^px_{2m})\|\rightarrow 0$ as $m,n\rightarrow\infty$. Thus $\{T^px_{2n}\}$ is a Cauchy sequence in $X$. Since $(X,d)is$ a complete cone metric spaces, there exist $u\in X$ Such that

$$\lim_{n\rightarrow\infty}T^px_{2n}=u\ldots\ldots\ldots\ldots \quad (3.2.7)$$

Since $T^p$is subsequently convergent, $\{x_{2n}\}$ has a convergent subsequence $\{x_{2m}\}$ such that

$$\lim_{m\rightarrow\infty}T^px_{2m}=T^pv\ldots\ldots\ldots\ldots \quad (3.2.8)$$

Since $T$ is injective, then by (3.1.8), we obtain

$$\lim_{m\rightarrow\infty}x_{2m}=T^pv.\ \ldots\ldots\ldots\ldots \quad (3.2.9)$$

By the uniqueness of the limit,

$$u = Tv \ldots\ldots \qquad \ldots (3.2.10).$$

Since $T_1$nd $T_2$ are continuous. So,

$\lim_{m\to\infty} T_2x_{2m} = T_2v$ and $\lim_{m\to\infty} T^p x_{2m} = T_1v$.Again ,Since$T$ is continuous, so, $\lim_{m\to\infty} T^pT_2x_{2m} = T^pT_2v$ and $\lim_{m\to\infty} T^pT_1x_{2m} = T^pT_1v$. Thus, if m is odd. Then,

$$\lim_{n\to\infty} T^pT_2x_{2n+1} = T^pT_2v \qquad (3.2.11)$$

So, now consider,

$$d(T^pT_2v, T^pv) \le d(T^pT_2v, T^px_{2n+1}) + d(T^px_{2n+1}, T^pv)$$
$$\le \lambda\left[d(T^pT_2v, T^pT_2x_{2n+1}) + d(T^px_{2n+1,}T^pT_1T_2v)\right] + d(T^px_{2n+1,}T^pv).$$
$$\le \lambda\left[d(T^pv, T^pv) + d(T^px_{2n+1,}T^px_{2n+2})\right] + d(T^px_{2n+1,}T^pv).$$

So, $d(T^pT_2v, T^pv) \le \frac{\lambda}{1-\lambda}\ \ d(T^px_{2n,}T^pv) + \frac{1}{1-\lambda} d\big(T^pv, T^px_{2n+1,}\big)$.

Since $P$ is normal cone with normal constant $K$.So, we get

$\|d(T^pT_2v, T^pv)\| \le K[\frac{\lambda}{1-\lambda}\ \ \|d(T^px_{2n,}T^pv)\| + \frac{1}{1-\lambda}\|d\big(T^pv, T^px_{2n+1,}\big)\|] \to 0$

Hence $\|d(T^pT_2v, T^pv)\| = 0$. This implies$T^pT_2v = T^pv$. Since T is injective. So,

$v = T_2v$ .Thus $v$ is the fixed point of$T_2$.

Similarly, it can be established that, $v$ is also fixed point of $T_1$, that means, $v$ is common fixed point of $T_1$and $T_2$.

**Now to prove uniqueness:** Suppose that $w$ is another common fixed point of $T_1$and $T_2$, then $T_1w = w = T_2w$.

Now, $d(T^Pv, T^pw) = d(T^pT_1v, T^PT_2w)$

$$\le \lambda[d(T^pv, T^pT_1v) + d(T^pw, T^pT_2w)]$$

$$d(T^Pv, T^pw) \le 0$$

Hence $d(T^Pv, T^pw) = 0$ which implies that, $T^pv = T^pw$. As $T$is injective, $v = w$ is the unique common fixed point of $T_1$and $T_2$. Since we have assumed that $\{T, T_1\}$ and $\{T, T_2\}$ are Banach pair,$\{T, T_1\}$ and $\{T, T_2\}$ Commutes at the fixed point of $T_1$and $T_2$, respectively. This implies that, $TT_1v = T_1Tv$ for $v \in F(T_1)$. So, $Tv = T_1Tv$, which gives that $Tv$ is another fixed point of $T_1$ It is also true for $T_2$.By the uniqueness of fixed point of $T_1$, $Tv = v$. Hence $v = Tv = T_1v = T_2v$. Therefore $v$ is the unique common fixed point of $T, T_1$and $T_2$ in $X$. This completes the proof of theorem.

**Theorem 3.3**: Let $(X, d)$be cone metric spaces and let $T, T_1, T_2 : X \to X$ be any three continuous self mappings on $X$. Assume that $T$ is an injective maps and $P$ is a normal cone with normal constant. If the mapping $T, T_1$ and $T_2$ satisfy the condition

$$d(T^pT_1x, T^pT_2y) \le \lambda[d(T^px, T^pT_2y) + d(T^py, T^pT_1x)] \ldots \ldots \qquad (3.3.1)$$

for all$x, y \in X$ where p is a positive integer and $\propto \in [0, \frac{1}{2})$. Then $T_1$ and $T_2$have an unique common fixed point in $X$. Moreover, if $(T, T_1)$ and $(T, T_2)$ are Banach pair, then $T, T_1$ and $T_2$ have a unique common fixed point in $X$.

**Proof:** Let $x_0$ be an arbitrary point in $X$. We define the iterative sequence$\{x_{2n}\}$ and $\{x_{2n+1}\}$by

$$x_{2n+1} = T_1x_{2n} = {T_1}^{2n}x_0 \ldots\ldots \qquad (3.3.2)$$

and

$$x_{2n+2} = T_2x_{2n+1} = {T_2}^{2n+1}x_0 \ldots\ldots \qquad (3.3.3).$$

Then $x = x_{2n}$ and $y = x_{2n-1}$from (3.1.1) we have

$$d(T^px_{2n+1,}T^px_{2n}) = d(T^pT_1\,x_{2n}, T^pT_2\,x_{2n-1})$$
$$\le \lambda[d\big(T^px_{2n,}T^pT_2x_{2n-1}\big) + d(T^px_{2n-1,}T^pT_2x_{2n})]$$
$$\le \lambda[d\big(T^px_{2n,}T^px_{2n+1}\big) + d(T^px_{2n-1,}T^px_{2n})]$$

So, $\quad d(T^px_{2n+1,}T^px_{2n}) \le \frac{\lambda}{1-\lambda}\, d(T^px_{2n-1,}T^px_{2n})$

$$\leq Ld(T^p x_{2n,} T^p x_{2n-1}) \ldots \quad (3.3.4)$$

Where $\frac{\lambda}{1-\lambda} = L < 1$.

So, for $m, n \in N$ with $n > m$ we have

$$d(T^p x_{2n,} T^p x_{2m}) \leq d(T^p x_{2n,} T^p x_{2n-1}) + d(T^p x_{2n-1,} T^p x_{2n-2}) + \ldots + d(T^p x_{2m+1,} T^p x_{2m})$$

$$\leq (L^{2n-1} + L^{2n-2} + \cdots \ldots \ldots \ldots \ldots \ldots + L^{2n}) d(T^p x_{1,} T^p x_0),$$

$$\leq \frac{L^{2n}}{1-L} d(T^p x_{1,} T^p x_0), \ldots \quad .(3.3.5)$$

Since $P$ is normal cone with normal constant, so by (3.1.5) we get

$$\|d(T^p x_{2n,} T^p x_{2m})\| \leq \frac{L^{2n}}{1-L} \|d(T^p x_{1,} T^p x_0)\| \ldots \quad .(3.3.6)$$

Since $k \in (0,1) \Rightarrow k \to 0 \; as \; n \to \infty$. Therefore $\|d(T^p x_{2n,} T^p x_{2m})\| \to 0$ as $m, n \to \infty$. Thus $\{T^p x_{2n}\}$ is a Cauchy sequence in $X$. Since $(X, d) is$ a complete cone metric spaces, there exist $u \in X$ Such that

$$\lim_{n\to\infty} T^p x_{2n} = u \ldots \ldots \quad .(3.3.7)$$

Since $T^p$is subsequently convergent, $\{x_{2n}\}$ has a convergent subsequence $\{x_{2m}\}$ such that

$$\lim_{m\to\infty} T^p x_{2m} = T^p v \ldots \ldots \quad (3.3.8)$$

Since $T$ is injective, then by (3.1.8), we obtain

$$\lim_{m\to\infty} x_{2m} = T^p v. \ldots \ldots \quad .(3.3.9)$$

By the uniqueness of the limit,

$$u = Tv \ldots \quad (3.3.10).$$

Since $T_1$nd $T_2$ are continuous. So,

$\lim_{m\to\infty} T_2 x_{2m} = T_2 v$ and $\lim_{m\to\infty} T^p x_{2m} = T_1 v$.Again ,Since$T$ is continuous, so, $\lim_{m\to\infty} T^p T_2 x_{2m} = T^p T_2 v$ and $\lim_{m\to\infty} T^p T_1 x_{2m} = T^p T_1 v$. Thus, if m is odd. Then,

$$\lim_{n\to\infty} T^p T_2 x_{2n+1} = T^p T_2 v \quad \ldots \quad (3.3.11)$$

So, now consider,

$$d(T^p T_1 v, T^p v) \leq d(T^p T_1 v, T^p x_{2n}) + d(T^p x_{2n}, T^p v)$$

$$\leq \lambda [d(T^p v, T^p T_2 x_{2n-1}) + d(T^p x_{2n-1,} T^p T_1 v)] + d(T^p x_{2n,} T^p v).$$

$$\leq \lambda \, [d(T^p v, T^p x_{2n}) + T^p d(T^p x_{2n-1,} T_1 v)] + d(T^p x_{2n,} T^p v).$$

$$\leq \lambda [d(T^p v, T^p x_{2n}) + d(T^p T_1 v, T^p v) + d(T^p x_{2n-1,} T_1 v)] + d(T^p x_{2n,} T^p v).$$

So, $d(T^p T_1 v, T^p v) \leq \frac{\lambda}{1-\lambda} \; [\, d(T^p x_{2n,} T^p v) + d(T^p x_{2n-1,} T_1 v)] + \frac{1}{1-\lambda} d(T^p v, T^p x_{2n})$.

Since $P$ is normal cone with normal constant $K$.So, we geT

$$\|d(T^p T_1 v, T^p v)\| \leq K[\frac{\lambda}{1-\lambda} \; (\| d(T^p x_{2n,} T^p v)\| + \|d(T^p x_{2n-1,} T_1 v)\|) \; + \frac{1}{1-\lambda} \|d(T^p v, T^p x_{2n+1,})\|] \to 0$$

Hence $\|d(T^p T_1 v, T^p v)\| = 0$. This implies$T^p T_1 v = T^p v$. Since T is injective. So,

$v = T_1 v$ .Thus $v$ is the fixed point of$T_1$.

Similarly, it can be established that, $v = T_2 v$ . Hence $T_1 v = v = T_2 v$ . $v$ is common fixed point of $T_1$and $T_2$.

**Now to prove uniqueness:** Suppose that $w$ is another common fixed point of $T_1$and $T_2$, then $T_1 w = w = T_2 w$.

Now, $d(T^P v, T^p w) = d(T^p T_1 v, T^P T_2 w)$

$$\leq \lambda [d(T^p v, T^p T_1 v) + d(T^p w, T^p T_2 w)]$$

$$d(T^P v, T^p w) \leq 0$$

Hence $d(T^P v, T^p w) = 0$ which implies that, $T^p v = T^p w$. As $T$is injective, $v = w$ is the unique common fixed point of $T_1$ and $T_2$

Since we have assumed that $\{T, T_1\}$ and $\{T, T_2\}$ are Banach pair,$\{T, T_1\}$ and $\{T, T_2\}$ Commutes at the fixed point of $T_1$and $T_2$, respectively. This implies that, $TT_1 v = T_1 T v$ for $v \in F(T_1)$. So, $Tv = T_1 Tv$, which gives that $Tv$ is another fixed point of $T_1$ It is also true for $T_2$.By the uniqueness of fixed point of $T_1$, $Tv = v$. Hence $v = Tv = T_1 v = T_2 v$. Therefore $v$ is the unique common fixed point of $T, T_1$and $T_2$ in $X$. This completes the proof of theorem.

**Theorem 3.4**: Let $(X, d)$be cone metric spaces and let $T, T_1, T_2 : X \to X$ be any three continuous self mappings on $X$. Assume that $T$ is an injective maps and $P$ is a normal cone with normal constant. If the mapping $T, T_1$ and $T_2$ satisfy the condition

$$d(T^p T_1 x, T^p T_2 y) \leq kd(T^p x, T^p T_2 y) + L\, d(T^p x, T^p y)\ldots \ldots\ldots \qquad (3.4.1)$$

for all$x, y \in X$ where p is a positive integer and $k, L \in [0,1)$ is constant. Moreover, if $(T, T_1)$ and $(T, T_2)$ are Banach pair, then $T, T_1$ and $T_2$ have a unique common fixed point in $X$. whenever $k + L < 1$.

**Proof:** Let $x_0$ be an arbitrary point in $X$. We define the iterative sequence$\{x_{2n}\}$ and $\{x_{2n+1}\}$by

$$x_{2n+1} = T_1 x_{2n} = T_1{}^{2n} x_0 \ldots\ldots \qquad (3.4.2)$$

and

$$x_{2n+2} = T_2 x_{2n+1} = T_2{}^{2n+1} x_0 \ldots\ldots \qquad (3.4.3).$$

Then $x = x_{2n}$ and $y = x_{2n-1}$from (3.1.1) we have

$$d(T^p x_{2n+1,} T^p x_{2n}) = d(T^p T_1\, x_{2n}, T^p T_2\, x_{2n-1})$$
$$\leq kd\big(T^p x_{2n,} T^p T_2 x_{2n-1}\big) + Ld(T^p x_{2n,} T^p x_{2n-1})$$
$$= Ld(T^p x_{2n,} T^p x_{2n-1})]$$

So, $d(T^p x_{2n+1,} T^p x_{2n}) \leq L^{2n}\, d(T^p x_{1,} T^p x_0) \ldots\ldots \qquad (3.4.4)$

So, for $m, n \in N$ with $n > m$ we have

$$d(T^p x_{2n,}\, T^p x_{2m}) \leq d\big(T^p x_{2n,} T^p x_{2n-1}\big) + d\big(T^p x_{2n-1,} T^p x_{2n-2}\big) + \ldots\ldots + d(T^p x_{2m+1,} T^p x_{2m})$$
$$\leq (L^{2n-1} + L^{2n-2} + \cdots \ldots\ldots\ldots\ldots\ldots + L^{2n}) d(T^p x_{1,} T^p x_0),$$
$$\leq \frac{L^{2n}}{1-L}\, d(T^p x_{1,} T^p x_0), \ldots\ldots \qquad (3.4.5)$$

Since $P$ is normal cone with normal constant, so by (3.1.5) we get

$$\left\|d(T^p x_{2n,}\, T^p x_{2m})\right\| \leq \frac{L^{2n}}{1-L} \left\|d(T^p x_{1,} T^p x_0)\right\| \ldots\ldots \qquad (3.4.6)$$

Since $k \in (0,1) \Rightarrow k \to 0$ $as$ $n \to \infty$. Therefore $\left\|d(T^p x_{2n,}\, T^p x_{2m})\right\| \to 0$ as $m, n \to \infty$. Thus $\{T^p x_{2n}\}$ is a Cauchy sequence in $X$. Since $(X, d)is$ a complete cone metric spaces, there exist $u \in X$ Such that

$$\lim_{n\to\infty} T^p x_{2n} = u \ldots\ldots \qquad (3.4.7)$$

The rest of proof is similar to the proof of theorem 3.1.

So, now consider,

$$d(T^p T_1 v, T^p v) \leq d(T^p T_1 v, T^p x_{2n}) + d(T^p x_{2n}, T^p v)$$
$$\leq kd(T^p v, T^p T_2 x_{2n-1}) + Ld(T^p v, T^p x_{2n-1}) + d(T^p x_{2n,} T^p v).$$
$$\leq kd(T^p v, T^p x_{2n}) + Ld(T^p v, T^p x_{2n-1}) + d(T^p x_{2n,} T^p v).$$

Since $P$ is normal cone with normal constant $K$.So, we get

$$\|d(T^p T_1 v, T^p v)\| \leq K[\ \|d(T^p x_{2n,} T^p v)\| + \|d(T^p x_{2n-1,} T_1 v)\| + \|d(T^p v, T^p x_{2n+1,})\|] \to 0$$

Hence $\|d(T^p T_1 v, T^p v)\| = 0$. This implies$T^p T_1 v = T^p v$. Since T is injective. So,

$v = T_1 v$ .Thus $v$ is the fixed point of$T_2$.

Similarly, it can be established that, $v = T_2 v$ . Hence$T_1 v = v = T_2 v$ . $v$ is common fixed point of $T_1$and $T_2$.

**Now to prove uniqueness:** Suppose that $w$ is another common fixed point of $T_1$and $T_2$, then $T_1 w = w = T_2 w$.

Now, $d(T^P v, T^p w) = d(T^p T_1 v, T^P T_2 w)$

$$\le kd(T^p v, T^p T_2 w) + Ld(T^p v, T^P w)$$
$$= kd(T^p v, T^p w) + Ld(T^p v, T^P w)$$
$$= (k + L)d(T^p w, T^p v).\text{Since } 0 \le k + l < 1$$

Thus, $\|d(T^P v, T^p w\| \le 0$. Hence $d(T^P v, T^p w) = 0$ which implies that, $T^p v = T^p w$. As $T$is injective, $v = w$ is the unique common fixed point of $T_1$and $T_2$.

Since we have assumed that $\{T, T_1\}$ and $\{T, T_2\}$ are Banach pair,$\{T, T_1\}$ and $\{T, T_2\}$ Commutes at the fixed point of $T_1$and $T_2$, respectively. This implies that, $TT_1 v = T_1 T v$ for $v \in F(T_1)$. So, $Tv = T_1 T v$, which gives that $Tv$ is another fixed point of $T_1$ It is also true for $T_2$.By the uniqueness of fixed point of $T_1$, $Tv = v$. Hence $v = Tv = T_1 v = T_2 v$. Therefore $v$ is the unique common fixed point of $T, T_1$and $T_2$ in $X$. This completes the proof of theorem.

**Theorem 3.5**: Let $(X, d)$be cone metric spaces and let $T, T_1, T_2 : X \to X$ be any three continuous self mappings on $X$. Assume that $T$ is an injective maps and $P$ is a normal cone with normal constant. If the mapping $T, T_1$ and $T_2$ satisfy the condition

$$d(T^p T_1 x, T^p T_2 y) \le pd(T^p x, T^p y) + q\, d(T^p x, T^p T_1 x) + rd(T^p y, T^p T_2 y)$$
$$+ s[d(T^p x, T^p T_1 x) + d(T^p y, T^p T_2 y)] + t[d(T^p x, T^p T_2 y) + d(T^p y, T^p T_1 x)] \text{ ……………(3.5.1)}$$

for all$x, y \in X$ where p is a positive integer and $k, l, m, n \in [0,1)$ is constant with$p + q + r + 2s + 2t < 1$. Moreover, if $(T, T_1)$ and $(T, T_2)$ are Banach pair, then $T, T_1$ and $T_2$ have a unique common fixed point in $X$. whenever $k + L < 1$.

**Proof:** Let $x_0$ be an arbitrary point in $X$. We define the iterative sequence$\{x_{2n}\}$ and $\{x_{2n+1}\}$by

$$x_{2n+1} = T_1 x_{2n} = {T_1}^{2n} x_0 \text{……………………….. …………………. } \quad (3.5.2)$$

and

$$x_{2n+2} = T_2 x_{2n+1} = {T_2}^{2n+1} x_0 \text{…………………. ………………… } \quad (3.5.3).$$

Then $x = x_{2n}$ and $y = x_{2n-1}$from (3.4.1) we have

$$d(T^p x_{2n+1,} T^p x_{2n}) = d(T^p T_1\, x_{2n}, T^p T_2\, x_{2n-1})$$
$$\le pd(T^p x_{2n,} T^p x_{2n-1}) + qd(T^p x_{2n,} T^p T_1 x_{2n}) + rd(T^p x_{2n-1,} T^p T_2 x_{2n-1})$$
$$+ s[d(T^p x_{2n,} T^p T_1 x_{2n}) + d(T^p x_{2n-1,} T^p T_2 x_{2n-1})]$$
$$+ t[d(T^p x_{2n,} T^p T_2 x_{2n-1}) + d(T^p x_{2n-1,} T^p T_2 x_{2n})]$$
$$\le pd(T^p x_{2n,} T^p x_{2n-1}) + qd(T^p x_{2n,} T^p x_{2n+1}) + rd(T^p x_{2n-1,} T^p x_{2n})$$
$$+ s[d(T^p x_{2n,} T^p x_{2n+1}) + d(T^p x_{2n-1,} T^p x_{2n})]$$
$$+ t[d(T^p x_{2n,} T^p x_{2n}) + d(T^p x_{2n-1,} T^p x_{2n+1})]$$
$$d(T^p x_{2n+1,} T^p x_{2n}) \le pd(T^p x_{2n,} T^p x_{2n-1}) + qd(T^p x_{2n,} T^p x_{2n+1}) + rd(T^p x_{2n-1,} T^p x_{2n})$$
$$+ s[d(T^p x_{2n,} T^p x_{2n+1}) + d(T^p x_{2n-1,} T^p x_{2n})]$$
$$+ t\,[d(T^p x_{2n+1,} T^p x_{2n}) + d(T^p x_{2n}, x_{2n-1})]$$
$$\le (q + s + t) d(T^p x_{2n,} T^p x_{2n+1}) + (p + r + s + t) d(T^p x_{2n-1,} T^p x_{2n})$$

This implies that,

$$d(T^p x_{2n+1,} T^p x_{2n}) \le \frac{(p+r+s+t)}{1-(q+s+t)}\, d(T^p x_{2n-1,} T^p x_{2n})$$
$$\le Ld(T^p x_{2n,} T^p x_{2n-1}), \text{ where } \frac{(p+r+s+t)}{1-(q+s+t)} = L < 1$$

Proceeding further,

So, $d(T^p x_{2n+1,} T^p x_{2n}) \le L^{2n}\, d(T^p x_{1,} T^p x_0)$…………………………………… (3.5.4)

So, for $m, n \in N$ with $n > m$ we have

$$d(T^p x_{2n,}\, T^p x_{2m}) \le d(T^p x_{2n,} T^p x_{2n-1}) + d(T^p x_{2n-1,} T^p x_{2n-2}) + \ldots + d(T^p x_{2m+1,} T^p x_{2m})$$

$$\leq (L^{2n-1} + L^{2n-2} + \cdots \ldots \ldots \ldots \ldots \ldots + L^{2n})d(T^p x_1, T^p x_0),$$

$$\leq \frac{L^{2n}}{1-L} d(T^p x_1, T^p x_0), \ldots \ldots \ldots \ldots \quad .(3.5.5)$$

Since $P$ is normal cone with normal constant, so by (3.1.5) we get

$$\|d(T^p x_{2n}, T^p x_{2m})\| \leq \frac{L^{2n}}{1-L} \|d(T^p x_1, T^p x_0)\| \ldots \ldots \quad (3.5.6)$$

Since $k \in (0,1) \Rightarrow k \to 0$ as $n \to \infty$. Therefore $\|d(T^p x_{2n}, T^p x_{2m})\| \to 0$ as $m, n \to \infty$. Thus $\{T^p x_{2n}\}$ is a Cauchy sequence in $X$. Since $(X, d) is$ a complete cone metric spaces, there exist $u \in X$ Such that

$$\lim_{n\to\infty} T^p x_{2n} = u \ldots \ldots \ldots \ldots \quad .(3.5.7)$$

The rest of proof is similar to the proof of theorem 3.1.

So, now consider,

$$\begin{aligned}
d(T^pT_1v, T^pv) &\leq d(T^pT_1v, T^px_{2n}) + d(T^px_{2n}, T^pv) \\
&\leq d(T^pT_1v, T^pT_2x_{2n-1}) + d(T^px_{2n}, T^pv) \\
&\leq pd(T^pv, T^px_{2n-1}) + qd(T^pv, T^pT_1v) + rd(T^px_{2n-1}, T^pT_2x_{2n-1}) \\
&\quad + s[d(T^pv, T^pT_1v) + d(T^px_{2n-1}, T^pT_2x_{2n-1})] \\
&\quad + t[d(T^pv, T^pT_2x_{2n-1}) + d(T^px_{2n-1}, T^PT_1v) + d(T^px_{2n}, T^pv) \\
&= pd(T^pv, T^px_{2n-1}) + qd(T^pv, T^pT_1v) + rd(T^px_{2n-1}, T^px_{2n}) \\
&\quad + s[d(T^pv, T^pT_1v) + d(T^px_{2n-1}, T^px_{2n})] \\
&\quad + t[d(T^pv, T^px_{2n}) + d(T^px_{2n-1}, T^PT_1v) + d(T^px_{2n}, T^pv) \\
&\leq pd(T^pv, T^px_{2n-1}) + (q+s)\, d(T^pv, T^pT_1v) + (r+s)\, d(T^px_{2n-1}, T^px_{2n}) \\
&\quad + (1+t)\, d(T^px_{2n}, T^pv) + td(T^px_{2n-1}, T^PT_1v)
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(T^pT_1v, T^pv) &\leq \frac{p}{1-(q+s)} d(T^pv, T^px_{2n-1}) + \frac{(r+s)}{1-(q+s)} d(T^px_{2n-1}, T^px_{2n}) + \frac{1+t}{1-(q+s)} d(T^px_{2n}, T^pv) \\
&\quad + \frac{t}{1-(q+s)} d(T^px_{2n-1}, T^PT_1v) \\
&\leq \frac{p}{1-(q+s)} [d(T^pv, T^px_{2n}) + d(T^px_{2n}, T^px_{2n-1})] + \frac{(r+s)}{1-(q+s)} d(T^px_{2n-1}, T^px_{2n}) \\
&\quad + \frac{1+t}{1-(q+s)} d(T^px_{2n}, T^pv) + \frac{t}{1-(q+s)} [d\big(T^px_{2n-1}, T^px_{2n}\big) + d(T^px_{2n}, T^pv) + d(T^pv, T^PT_1v)]
\end{aligned}$$

Hence

$$d(T^pT_1v, T^pv) \leq \frac{1+p+2t}{1-q-s-t} d(T^pv, T^px_{2n}) + \frac{p+r+s+t}{1-q-s-t} d\big(T^px_{2n-1}, T^px_{2n}\big)$$

Since $P$ is normal cone with normal constant $K$.So, we get

$$\|d(T^pT_1v, T^pv)\| \leq K[\frac{1+p+2t}{1-q-s-t} \|d\big(T^px_{2n}, T^pv\big)\| + \frac{p+r+s+t}{1-q-s-t} \|d\big(T^px_{2n-1}, T_1v\big)\|] \to 0.$$

Thus, $\|d(T^pT_1v, T^pv)\| = 0$. This implies $T^pT_1v = T^pv$. Since T is injective. So,

$v = T_1v$ .Thus $v$ is the fixed point of $T_2$.

Similarly, it can be established that, $v = T_2v$ . Hence $T_1v = v = T_2v$ . $v$ is common fixed point of $T_1$ and $T_2$.

**Now to prove uniqueness:** Suppose that, if $w$ is another common fixed point of $T_1$ and $T_2$, then $T_1w = w = T_2w$.Now consider

$$\begin{aligned}
d(T^Pv, T^pw) &= d(T^pT_1v, T^PT_2w) \\
&\leq pd(T^pv, T^pw) + qd(T^pv, T^pT_1v) + rd(T^pw, T^pT_2w) \\
&\quad + s[d(T^pv, T^pT_1v) + d(T^pw, T^pT_2w)] + t[d(T^pv, T^pT_2w) + d(T^pw, T^pT_1v)] \\
&\leq pd(T^pv, T^pw) + 2td(T^pv, T^pw) \\
&= (p+2t)d(T^pv, T^pw) \\
&\leq (p+q+r+s+t)d(T^pv, T^p)
\end{aligned}$$

$< \ d(T^p v, T^p w)$ as $p + q + r + s + t < 1$.

Thus, $\|d(T^P v, T^p w\| \leq 0$. Hence $d(T^P v, T^p w) = 0$ which implies that, $T^p v = T^p w$. As $T$is injective, $v = w$ is the unique common fixed point of $T_1$and $T_2$.

Since we have assumed that $\{T, T_1\}$ and $\{T, T_2\}$ are Banach pair,$\{T, T_1\}$ and $\{T, T_2\}$ Commutes at the fixed point of $T_1$and $T_2$, respectively. This implies that, $TT_1 v = T_1 T v$ for $v \in F(T_1)$. So, $Tv = T_1 T v$, which gives that $Tv$ is another fixed point of $T_1$ It is also true for $T_2$.By the uniqueness of fixed point of $T_1$, $Tv = v$. Hence $v = Tv = T_1 v = T_2 v$. Therefore $v$ is the unique common fixed point of $T, T_1$and $T_2$ in $X$. this completes the proof of theorem.

## 4. Conclusion

In this attempt, we generalize unique common fixed point results in complete cone metric spaces with two Banach pairs mapping satisfying contraction condition given by the concept of [44].These results generalize improve and extend the theorem 3.2,3.3,3.4, 3.5 which is given by Petwal &Dimri [41] of theorem 3.1.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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