A common Fixed Point Theorem in Generalised Partially Ordered G-metric space For Weak C-contraction mappings using Ω-distance.

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Abstract:

In this paper, we prove a common fixed point theorem on a complete, generalized Partially Ordered G-metric space for a weak C- contraction mapping using the concept of

Ω-distance.Mustafa and Sims introduced the concept of G-Metric space in the year 2004 as a generalization of metric spaces. In this type of spaces, a non-negative real number is assigned to every triplet of elements. Several other studies related to metric spaces are being extended to G-Metric spaces. Saadati et. Alex tended Ω-distanceTo G-metric spaces. Also we can find, the fixed point results in G-metric spaces be applied in the existence of solutions for a class of integral equations.

Recently Nieto and Rodriguez-Lopez, Ran and Reuriegs and Petrusel and Rus presented some new results for contractions in partially ordered metric spaces.

Binayak S.choudharyintroduced the class of weakly c-contractive mappings. He also established that these mappings have unique fixed points in complete metric space. By using the concept of G-metric space, Ω-distance is defined which is a generalization of the concept of ω-distance due to Kada, Suzuki and Takahashi

In this Research Paper, we prove a common fixed point theorem on a partially ordered complete G-metric space for a weak C-contraction mapping using the concept of Ω-distance.

Key words: G-metric spaces, Ω-distance, weak C-contractions, partially ordered G- metric space.

Introduction and preliminaries:

Mustafa and Sims [20] introduced the concept of G-metric spaces in the year of 2004 as a generalization of metric spaces. In thesetypesof spaces a non-negative real number is assigned to every triplet of elements. Many other studies related to metric spaces are being extended toG-metric spaces .In [27] Saadati et.al extended Ω-distance to G-metric spaces. Also we can find the fixed pint results in G-metric spaces be applied in the existence of solutions for a class of integral equations.

Definition 1:Partialy Ordered G-metric space:

Let$(X,\preccurlyeq)$be a Partially ordered set and let $G : X \times X \times X \to [0,\infty)$be a function satisfying the following:

i) $G(x,y,z) = 0$ if $x=y=z$(coincidence)
ii) $G(x,x,y) > 0$ for all $x,y \in X$,with $x \neq y$
iii) $G(x,x,z) \leq G(x, y,z)$ for all $x,y \in X$,with $z \neq y$
iv) $G(x,y,z)= G(x,z,y)=G(y,z,x)$ (symmetry)
v) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$.

Definition 2: G convergent

Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence of points in X.Then $\{x_n\}$ is called G-convergent to x if $\lim_{n\to\infty} G(x, x_n, x_m) = 0$, that is for each $\epsilon > 0$, there exists a positive integer N such that G$(x, x_n, x_m) < \epsilon$ for all m,n $\geq N$.Then we call x as the limit of the sequence and it is written as $\lim_{n\to\infty} x_n = x \; or \; x_n \to x$.

Definition 3: G- convergent

Let (X,G) be a G-metric space, then the following are equivalent:

i) $\{x_n\}$ is G-convergent to x.
ii) G$(x_n, x_n, x) \to 0$ as $n \to \infty$
iii) G$(x_n, x, x) \to 0$ as $n \to \infty$
iv) G$(x_m, x, x) \to 0$ as $n \to \infty$

Definition 4: G-cauchy sequence

i) Let (X,G) be a G-metric space . A sequence $\{x_n\}$ in X is said to be G-Cauchy sequence if , for each $\epsilon > 0$, there exists a positive integer n_0 such that for all m,n,l $\geq n_0$, G$(x_n, x_m, x_l) < \epsilon$
ii) A sequence $\{x_n\}$ in X is said to be G- convergent to a point x$\in X$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that for all m,n $\geq n_0$,

G$(x_n, x_m, x) < \epsilon$

Definition 4: C- Contraction:

Let (X,G) be a G-Metric space and T: $X \to X$ be a self mapping on (X,G).

Now T is said to be a C-contraction if there exists $\alpha\epsilon \left(0, \frac{1}{2}\right)$such that for all x, y,z $\in X$,

G (Tx, Ty,Tz)$\leq \alpha[G(x, Ty, Tz) + G(y, Tx, Tz) + G(z, Tx, Ty)]$

Definition 5: Contraction:

Let (X,d) be a G-Metric space and T: $X \to X$ be a self mapping on (X,G).Now T is said to be a contraction if G(Tx, Ty, Tz)$\leq \alpha G(x, y, z)$

For all x, y, z $\in X$ where 0$\leq \alpha < 1$.

Definition 6:Ω-distance

Let (X,G) be a G-metric space. Then a function Ω: X x X x X→ [0,∞) is called an Ω-distance on X if the following conditions are satisfied.

i) $\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z)$for all x,y,z,a ∈ X
ii) for any x, y ∈ X , $\Omega(x, y, .), \Omega(x, ., y)$:X →[0,∞) are lower semi-continuous,
iii) for each $\epsilon > 0$,there exists a $\delta > 0$ such that $\Omega(x, a, a) \leq \delta$ and $\Omega(a, y, z) \leq \delta$ imply G(x,y,z) $\leq \epsilon$

Definition 6: Weak C-contraction:

A mapping T:X→ X,where (X,G) is a metric space and Ω be an Ω- distance is said to be a weakly C-contractive or a weak C-contraction if for all x,y,$\omega \in X$,

$$\Omega(Tx, T^2x, T\omega) \leq \frac{1}{2}[\Omega(x, Tx, T\omega) + \Omega(\omega, Tx, Tx)] - \boldsymbol{\psi}[\Omega(x, Tx, T\omega) + \Omega(\omega, Tx, Tx)]$$

where $\boldsymbol{\psi} : [0, \infty)^2 \to [0, \infty)$ is a continuous mapping such that $\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$=0 ifonly if x=y=z=0.

Definition 7:E.A.Property:

Let A and S be two self mappings of a metric space(X,d).The pair (A,S) is said to satisfy Property

(E.A.) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to} Ax_n = \lim_{n\to} Sx_n = z$, for some z$\epsilon X$.

This property is mainly used to prove common fixed point theorems.

Lemma:

Let(X,G) is a metric space and Ω be an Ω- distance on X. Let $\{x_n\}$,$\{y_n\}$ be sequences in X. $\{\alpha_n\}$,$\{\beta_n\}$ be sequences in $[0,\infty)$ $converging$ to zero and let x,y,z,a $\in X$.Then the following hold:

i) If $\Omega(y, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y, z) \leq \beta_n$,for n $\in N$,then G(y,y,z)< ϵ and hence y=z.

ii) If $\Omega(\mathrm{y_n}, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, x_m, z) \leq \beta_n$,for any m>n $\in N$,then G($\mathrm{y_n}$, $\mathrm{y_m}$,z)→ 0 and hence $\mathrm{y_n} \to z$.

iii) If $\Omega(\mathrm{x_n}, x_m, x_l) \leq \alpha_n$, for anyl, m,n $\in N$,with $n \leq$ m $\leq$ l,then $\{x_n\}$ is a G-cauchy sequence.

iv) If $\Omega(\mathrm{x_n}, a, a) \leq \alpha_n$, for any n $\in N$, then $\{x_n\}$ is a G-cauchy sequence.

Proof:

To prove ii),let $\epsilon > 0$ be given. From the definition ofΩ- distance, there exists a$\delta > 0$ such that $\Omega(\mathrm{u}, a, a) \leq \delta$ and $\Omega(\mathrm{a, v, z}) \leq \delta$ imply G(u,v,z) $\leq \epsilon$.

Choose $n_0 \in N$such that $\alpha_n \leq \delta$ and $\beta_n \leq \delta$ for every n≥ n_0.Then for any m>n>n_0, $\Omega(\mathrm{y_n}, x_n, x_n) \leq \alpha_n \leq \delta$, $\Omega(x_n, y_m, z) \leq \beta_n \leq \delta$,

Hence G($\mathrm{y_n}$, $\mathrm{y_m}$,z) $\leq \epsilon$,so that $\{y_n\}$ converges to z.

To prove iii) let $\epsilon > 0$ be given, then choose $\delta > 0$ and then $n_0 \in N$.

Then for any l≥ $m > n \geq n_0$, $\Omega(x_n, x_{n+1}, x_{n+1}) \leq \alpha_n \leq \delta$, $\Omega(x_{n+1}, x_m, x_n) \leq \alpha_{n+1} \leq \delta$, and so $G(\mathrm{x_n}, x_m, x_l) \leq \epsilon$,⇒$\{x_n\}$ is a G-Cauchy sequence.

Condition iv) is the special case of iii).

From [11] Mustafa and Sims proved that the function G(x,y,z) is jointly continuous in all the three variables' is said to be Ω-bounded if there exists a constant M such that $\Omega(\mathrm{x, y, z}) \leq M$for all x,y,z $\in X$.

Main Result:

Theorem 2.1

Let (X,≤,G) be a partially ordered complete G-metric space.Suppose that there exists an Ω- distance on X and T is non-decreasing mapping from X→ X. Let X be Ω-bounded.

If there exists a weak C- contraction such that

$$\Omega(Tx, T^2x, T\omega) \leq \frac{1}{2}\{(\Omega(x, Tx, T\omega) + (\Omega(\omega, Tx, Tx)\} - \boldsymbol{\psi}[(\Omega(x, Tx, T\omega), (\Omega(\omega, Tx, Tx)]$$

for all x $\leq Tx$and $\omega \in X$.Also for every x $\in X$.

inf{$(\Omega(x, y, x) + (\Omega(x, y, Tx) + (\Omega(x, Tx, T\omega) + (\Omega(x, T^2x, y{:}\, x \leq Tx)\} > 0$ for every y $\in X$ with y≠ Ty.If there exists an $x_0 \in X$ with $x_0 \leq Tx_0$,then T has a fixed point.More over if v=Tv then **Ω**(v,v,v)=0.

Proof:

Let x_1 be the common fixed point of T.

To prove that $\mathrm{T}x_1 = x_1$.

Let us suppose that $\mathrm{T}x_1 \neq x_1$, then given that T is non-decreasing ,

$$x_1 \leq Tx_1,$$

$$x_1 \leq Tx_1 \leq T^2x_1 \leq \cdots \leq T^{n+1}x_1 \leq \cdots$$

For all n ∈ N and m≥ 0,

$(\Omega(T^n x_1, T^{n+1}x_1, T^{n+m}x_1) \leq \frac{1}{2}[\Omega(T^{n-1}x_1, T^n x_1, T^{n+m}x_1) + \Omega(T^{n+m-1}x_1, T^n x_1, T^n x_1)]$-

$\boldsymbol{\psi}[\Omega(T^{n-1}x_1, T^n x_1, T^{n+m}x_1), \Omega(T^{n+m-1}x_1, T^n x_1, T^n x_1)]$

$$\leq \frac{1}{2}[\Omega(T^{n-1}x_1, T^n x_1, T^{n+m}x_1) + \Omega(T^{n+m-1}x_1, T^n x_1, T^n x_1)] -$$

$\boldsymbol{\psi}\Big[\Omega(T^{n-1}x_1, T^{n+1}x_1, T^{n+1}x_1) + \Omega(T^{n+1}x_1, T^n x_1, T^{n+m}x_1), \Omega\left(T^{n+m-1}x_1, T^{n+1}x_1, T^{n+1}x_1\right) + \Omega(T^{n+1}x_1, T^n x_1, T^n x_1)\Big)\Big]$

$$\leq \frac{1}{2}[\Omega(T^{n-1}x_1, T^n x_1, T^{n+m}x_1) + \Omega(T^{n+m-1}x_1, T^n x_1, T^n x_1)] - \alpha_n$$

$$\leq \frac{1}{2}\alpha_n.$$

Therefore $\{T^n x_1\}$ is a Cauchy sequence.

Since X is a complete G-metric, $\{T^n x_1\}$ converges to $x_0 \in X$.

Let n∈ N be fixed.

Then by lower semi continuity of Ω we have for t>n,

$\Omega\,(T^n x_1, T^t x_1, x_0) \leq \lim_{z\to\infty} \inf \Omega\,(T^n x_1, T^t x_1, T^z x_1) \leq \delta$

And for $l \geq n$,

$\Omega\,(T^n x_1, x_0,\ T^l x_1) \leq \lim_{z\to\infty} \inf \Omega\,(T^n x_1, T^t x_1, T^z x_1) \leq \delta$

To prove uniqueness:

Let us assume that $z \neq Tz$.

We know that, $T^n x_1 \leq\ T^{n+1}x_1$,

∴ $0 \leq \inf \Omega\,(T^n x_1, z, T^n x_1) + \Omega\,(T^n x_0, z, T^{n+1}x_1) + \Omega(T^n x_0, T^{n+2}x_0, z)$ for n ∈ W.

$\leq \inf \{\ 3\ \delta : n \in\ N\}$

$= 0$

Which is a contradiction to our assumption that $z \neq Tz$.

∴ $z = Tz$.

If v=Tv, $\Omega(v, v, v) = \Omega(Tv, T^2v, T^3v)$

$\leq \frac{1}{2}[\ \Omega(v, Tv, T^2\ v) + \Omega(v, Tv, Tv)] - \boldsymbol{\psi}[\Omega(v, Tv, T^2\ v), \Omega(v, Tv, Tv)]$

$$\leq \frac{1}{2}[\ \Omega(v, Tv, T^2\ v) + \Omega(v, Tv, Tv)]$$

$= 0$

∴ $\Omega(v, v, v) = 0$.

Applications:

As an application of omega distance, we find the existence of solution of non-linear fractional differential inclusion with non-convex valued functions. Also in various fields of sciences such as physics, engineering, bio-physics, fluid dynamics etc.

Conclusion:

In this research attempt, we prove a common fixed point theorem in partially ordered complete-metric spaces with applications in various fields. This result generalizes the results of in the aim of employing omega distances and its contractive conditions.

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