A NEW METHOD OF SEPARATION OF VARIABLES AND IT'S APPLICATIONS

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Abstract: In this article a simpler and widely applicable analytical method of solution is presented to shallow water wave guide problems. In this article the method of separation of variables is proved to be applicable to solve waveguide problems with variable boundaries and medium. Applications of the method to solve waveguide problems in acoustic, radio and optical frequencies and more general models of waveguides are also explained. The solution by perturbation methods and the method of separation of variables presented does not express the frequency spectrum of the acoustic signal in the explicit manner. Therefore, a simple and widely applicable method of solution is presented in the 4^{th} section to such wave guide problems which express the frequency spectrum of the acoustic signal very explicitly.

Key words: **Waveguides with surface waves, Perturbation methods, frequency spectrum and Method of separation of variables.**

I. INTRODUCTION

A.H. Nayfeh developed perturbation methods to study sound propagation in acoustic waveguides with wavy surfaces [1 2]. Many models of oceanic wave guide had been built and many theories were developed to study the propagation of sound waves in shallow water oceanic wave guides. G.V. Anand and M.K. George [3] and V. Sundaravadivel [4] had studied the propagation of sound waves in simple oceanic wave guides with surface waves using analytical methods. They used perturbation methods developed by A.H.Nayfeh [1 2] which are applicable only to wave guides with low frequency and smaller amplitude surface waves. Moreover, perturbation method of solution generates singularity in the solution and complicates the determination of solution at points close to the singularity. Therefore, later on John Daniel [5 6 7 8] developed a simpler method of solution using surface wave phase modulation method which is applicable for any value of surface wave frequency and amplitude and without any singularity in the solution. In this article, a much simpler analytical solution based on the method of separation of variables is presented which is also applicable for any value surface wave amplitude and frequency without any singularity in the solution. In the last section, a method of determination of frequency spectrum of acoustic signals is also suggested.

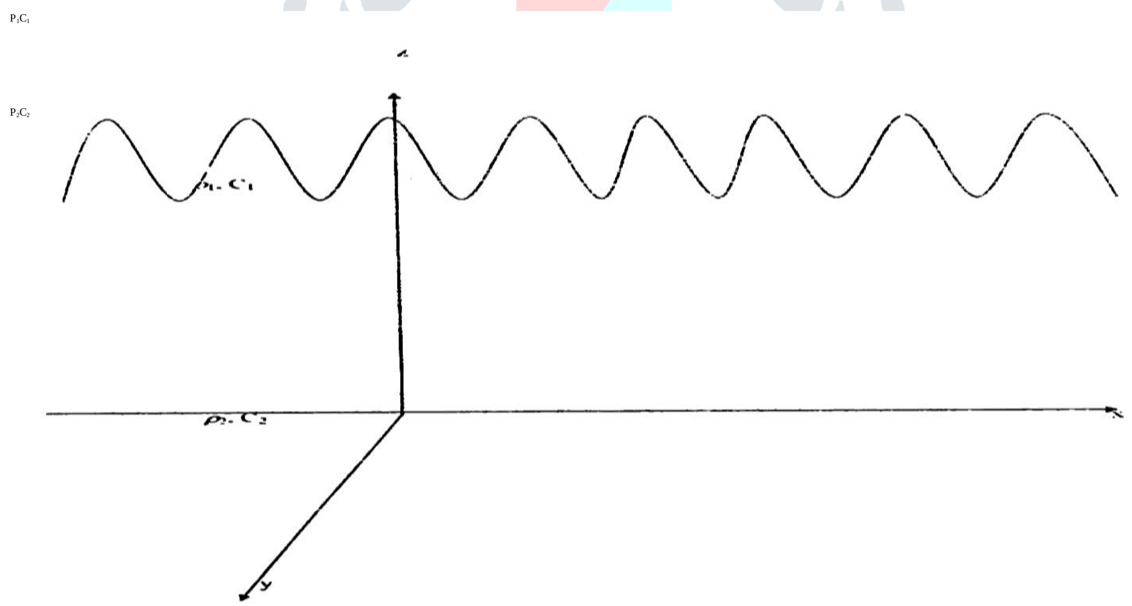

Figure – 1

II. THEORY

Consider an oceanic wave guide with a wavy surface as shown in the Figure-1. The surface wave is a single frequency wave propagating in x direction which can be expressed mathematically as

$$Z = h(x, t) = h_0 + a \operatorname{Cos}(\alpha x - \Omega t) \tag{1}$$

Where h_0 is average channel depth, α is wave number, Ω is circular frequency and a is amplitude of the surface wave. Let ρi, Ci (i = 1,2) denote respectively the density and velocity of sound in the two media. Medium 2 is assumed to be a semi-infinite one. The interface between medium 1 and medium 2 is assumed to be flat.

Let us assume that a plane sound wave propagates in the wave guide in the direction x. The acoustic pressure P (x, y, z, t) can be determined by solving the wave equation.

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}-\frac{1}{C^2}\frac{\partial^2 P}{\partial t^2}=0 \qquad (2)$$

with boundary conditions

P (x, y, z, t) = 0 at z = h (x, t) (3a)

P(x , y, + 0, t) = P (x, y, -0, t) (3b)

$$Vz\Big|_{z=+0} = Vz\Big|_{z=-0} = \frac{1}{\rho_1}\frac{\partial P}{\partial z}\Big|_{z=+0} = \frac{1}{\rho_2}\frac{\partial P}{\partial z}\Big|_{z=-0} \qquad (3c)$$

where Vz is z component of particle velocity

P → 0 a z → - ∞ (3d)

Where +0 and -0 indicates that the interface z = 0 is approached from the sides z > 0 and z < 0 respectively.
Since there is no variation of pressure in y direction, equation (2) can be written as

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}-\frac{1}{C^2}\frac{\partial^2 P}{\partial t^2}=0 \qquad (4)$$

Let us assume that Ω << ω where ω is angular frequency of sound wave. Therefore, equation (4) can be written as

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}+k^2 P \approx 0 \qquad (5)$$

Where

$$k=\frac{\omega}{c_1}=k_1 \;\; for\; 0<z<h \qquad (6a)$$

$$=\frac{\omega}{c_2}=k_2 \;\; for \;\; z<0 \qquad (6b)$$

Therefore, by separation of variables methods,

P (x, z, t) = Ψ_n (z).P_x(x) (7a)

$P_x(x)$ = Sin(ζ_n.x) (7b)

The function $\Psi_n(z)$ must satisfy the equation

$$\frac{\partial^2 \Psi_n}{\partial z^2}+(K^2-\xi_n^2\,(x,t))\,\Psi_n=0 \qquad (8)$$

with the boundary conditions

ψ_n = 0 at z = h (x, t) (9a)

ψ_n (+0, x, t) = ψ_n (-0, x, t) (9b)

$$\frac{1}{\rho_1}\frac{\partial \Psi_n}{\partial z}(+0,x,t)=\frac{1}{\rho_2}\frac{\partial \Psi_n}{\partial z}(-0,x,t) \qquad (9c)$$

Ψ_n → 0 as z → -∞ (9d)

The solution to the equation (8) is

Ψ_n = N_n sin χ_n (z/h – 1) for 0 < z < h

= Cn e^{Dnz} for z < 0 (10)

Where N_n, C_n, χ_n & D_n, are functions of x, t, Ψ_n must satisfy the orthonormality condition.

$$\int_{-\infty}^{h} \rho(z)^{-1} \psi_n \ \psi_m \ dz = \delta_{mn}$$ (11)

Where $\rho(z) = \rho_1$ for $0 < z < h$
$= \rho_2$ for $0 < z < h$
and δ_{mn} is kronecker delta.

By substituting the equating (10) into equation (1) and (8) we get

$$\int_{0}^{h} N_n^2 \rho_1^{-1} \sin^2 \chi_n (\frac{z}{h} - 1)\, dz + \int_{-\infty}^{h} \rho_2^{-1} C_n^2 e^{2Dnz} dz = 1$$ (12)

$$\xi_n^2 = K_1^2 - \left[\frac{\chi_n}{h}\right]^2$$ (13)

$$D_n = (\xi_n^2 - K_2^2)^{1/2}$$ (14)

Substitution of equation (10) into equations (9b) & (9c) gives,
$-N_n \sin\chi_n = C_n$ (15)

$$\frac{1}{\rho_1} N_n \left(\frac{\chi_n}{h}\right) \mathrm{Cos}\chi_n = C_n \frac{D_2}{\rho_2}$$ (16)

Equations (13) to (16) can be combined to get
$\mathrm{Cot}\ \chi_n = -(q\ \chi_n)^{-1}\ (h^2(K_1^2 - K_2^2) - \chi_n^2)^{1/2}$ (17)

where $q = \rho_2 / \rho_1$ From equation (12) we get

$$N_n = \sqrt{\frac{2}{h\rho_1^{-1}(1 - \frac{\sin 2\chi_n}{2\chi_n}) + \frac{\rho_1^{-1} \sin^2 \chi_n}{D_n}}}$$ (18)

χ_n can be found by solving the equation (17).

Thus the complete analytical solution of the problem is obtained. For $C_n = 0$, $\Psi_n = 0$ at $z = 0$. Therefore, $\sin \chi_n = n\pi$, where n is a positive integer. The solution is $P(x, z, t) = \sin \chi_n (z/h - 1). \mathrm{Sin}(\zeta_n x - \omega t)$. Therefore, the acoustic signal is phase modulated in the direction of z and x by the surface wave. In addition there will be an amplitude modulation of normal mode acoustic signal by the surface wave.

If the condition $\Omega << \omega$ is not satisfied, then the solution can be obtained by Fourier transforming $P_t(t)$, if $P(x, z, t) = P_x(x).P_z(z).P_t(t)$ is assumed as the solution. Then by inverse Fourier transforming $P_\omega(\omega)$ where $P_\omega(\omega)$ is Fourier transformation of $P_t(t)$. Therefore, $P_t(t)$ will be directly proportional to $\int 1/\omega^2.e^{j\omega t}\, d\omega$, ω varies from 0 to 2π. Acoustic signal is phase modulated in t dimension also by the surface waves.

The perturbation method used by Anand, et al and Sundaravadivel creates singularity. Therefore, accurate determination of field close to the singularity becomes very complicated. The method presented in this article does not generate any such singularity. Since the solution is valid for a generalized waveguide dimensions, the solution could be easily simplified to waveguides with smaller surface variations and frequencies by using Taylor series expansion and approximations. The solution is applicable to non periodic variations also.

III. APPLICATION OF THE METHOD

A.H.Nayfeh and O.H.Kandil [9] studied wave propagation in a circular cylindrical waveguide with sinusoidal boundary walls by applying perturbation techniques. The method presented in this article could be extended to study the wave propagation in such cylindrical waveguides with variable boundary. The boundary need not be periodic to apply the method. Similarly the method is applicable to study the wave propagation in the waveguides of A.M.Nusayr and M.A. Hawwa [10 11]. K.Srivatsava and Fabrizio Frezza, et al [12 13] had studied the wave propagation in 1-dimensional and two dimensional periodic waveguides. Both the variable boundary and the medium generate periodic waves in the propagating waves of waveguide without any boundary or medium variations. Therefore, variations of the medium could be equivalently transformed to variations in the boundary of the waveguide. After such transformation, the method developed in this article could be extended to find the field in the waveguide. More general waveguide models [14] needs numerical solution to the analytical expressions derived in this article.

IV. DETERMINATION OF FREQUENCY SPECTRUM

The sound wave which propagates in the wave guide without a surface wave will be phase modulated by the surface wave if the surface wave is present on the flat wave guide. This phase modulation will be a periodic function of x, t propagating in the x direction. Therefore, this phase function can be represented by

$$\varphi(x,t) = \sum_{n=0}^{\infty} A_n \mathrm{Cos}(n\,\alpha x - \Omega t) + \sum_{n=0}^{\infty} B_n \mathrm{Sin}(n\alpha\, x - \Omega t) \qquad (19)$$

The propagating periodic phase function can be easily found using ray tracing method. If the periodic phase function is known the constants A_n and B_n can be easily determined using Fourier theorem. The surface wave modulates the amplitude and the phase angles of the modes. Therefore,

$$P(x, z, t) = (E_n + \lambda_n.\ \Phi(x, t)).\Psi_n(z + \lambda_1.\varphi(x, t)).\ \cos(\square n\, x - \omega t + \lambda_2.\varphi(x, t)) \qquad (20)$$

Where E_n, λ_n, λ_1 and λ_2 are constants and the function $\Psi_n(z)$ must satisfy the wave equation (5)

If the condition $\Omega << \omega$ is not satisfied, then the solution can be obtained by Fourier transforming $P_t(t)$, if $P(x, z, t) = P_x(x).P_z(z).P_t(t)$ is assumed as the solution. Then by inverse Fourier transforming $P_\omega(\omega)$ where $P_\omega(\omega)$ is Fourier transformation of $P_t(t)$. Therefore, $P_t(t)$ will be directly proportional to $\int 1/\omega^2.e^{j\omega t}\, d\omega$, ω varies from 0 to 2π. Acoustic signal is phase modulated in t dimension also by the surface waves.

The perturbation method used by Anand, et al and Sundaravadivel creates singularity. Therefore, accurate determination of field close to the singularity becomes very complicated. The method presented in this section does not generate any such singularity.

V. CONCLUSION

A much simpler analytical solution to the problem of sound wave propagation in an ocean with a wavy surface is derived which is applicable to surface wave of any amplitude and frequency without any singularity in the solution. The method of solution could be extended easily to 2 and 3 dimensions also. The method could be extended to analyze the ionospheric radio wave propagation and also to analyze the light wave propagation in optical waveguides with surface irregularities.

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