Propeties Of Semi-I-Bitopological Groups Via Ideals

B. BRUNDHA* & P. GOMATHI SUNDARI**
** Department of Mathematics, Government Arts college for women, Orathanadu, Thanjavur, Tamilnadu*
*** Department of Mathematics, Rajah Serfoji Government College, Thanjavur, Tamilnadu.*

Abstract: In this paper, we introduce and study a class of ideal bitopologized groups called *(i, j)* - semi-I-bitopological groups.

I. INTRODUCTION

One of the important and basic topics in the theory of classical point set topology and several branches of Mathematics, which have been researched by many authors, is continuity of functions. Various types of continuous functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good many of them have been extended to the setting of multifunction. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [6] and Vaidyanathaswamy, [11]. An ideal I on a topological space $(X,\boldsymbol{\tau})$ is a nonempty collection of subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space $(X, \boldsymbol{\tau})$ with an ideal I on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, called the local function [11] of A with respect to $\boldsymbol{\tau}$ and I, is defined as follows: for $A \subset X$, $\mathbf{A}^\star(\boldsymbol{\tau}, I) = \{x \in X | U \cap A \notin I$ for every $U \in \boldsymbol{\tau}(x)\}$, where $\boldsymbol{\tau}(x) = \{U \in \boldsymbol{\tau} | x \in U\}$. A Kuratowski closure operator $\mathbf{Cl}^\star(.)$ for a topology $\boldsymbol{\tau}^\star(\boldsymbol{\tau}, I)$ called the *-topology, finer than $\boldsymbol{\tau}$ is defined by $\mathbf{Cl}^\star(A) = A \cup \mathbf{A}^\star(\boldsymbol{\tau}, I)$ when there is no chance of confusion, $\mathbf{A}^\star(I)$ is denoted by $\mathbf{A}^\star$. If I is an ideal on X, then $(X, \boldsymbol{\tau}, I)$ is called an ideal topological space. If $(G, *)$ is a group, and τ_1 and τ_2 are topologies on G, then we say that $(G, *, \tau_1, \tau_2, I)$ is a bitopologized group. Given a bitopologized group G, a question arises about interactions and relations between algebraic and bitopological structures : which topological properties are satisfied by the multiplication mapping $m : G \times G \rightarrow G, (x, y) \rightarrow x * y$, and the inverse mapping $i : G \rightarrow G, x \rightarrow x^{-1}$. In this paper, we introduce and study a class of bitopologized groups called *(i, j)*-semi-I-bitopological groups.

II. PRELIMINARIES:

Throughout this paper $(G,*, \tau_1, \tau_2, I)$, or simply G, will denote a group $(G, *)$ endowed with the topologies τ_1 and τ_2 on G. The identity element of G is denoted by e, or e_G when it is necessary, the operation $* : G \times G \rightarrow G, (x, y) \rightarrow x * y$, is called the multiplication mapping and sometimes denoted by m, and the inverse mapping $i : G \rightarrow G, x \rightarrow x^{-1}$ is denoted by i. For a subset A of a topological space (X, τ_i), $i\ Cl(A)$ and $i\ Int(A)$ denote the closure of A and interior of A in (X, τ_i), respectively.

Definition 2.1: [2] A subset S of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-semiopen if $S \subset j\ Cl^*(i\ Int(S))$. The complement of an (i, j)-semi-I-open set is called an (i, j)–semi-I-closed set.

Definition 2.2: [2] The intersection of all (i, j)–semi-I-closed sets containing $S \subset X$ is called (i, j)–semi-I-closure of S and is denoted by (i ,j)-s Cl (S). The family of all (i, j)–semi-I-open (resp. (i ,j)–semi-I-closed) sets of (X, τ_1, τ_2) is denoted by (i, j)–SIO (X) (resp. (i, j)-SIC (X)). The family of all (i, j)–semi-I-open (resp. (i, j)–semi-I-closed) sets of (X, τ_1, τ_2) containing a point $x \in X$ is denoted by (i, j)–SIO (X, x) (resp. (i, j)–SIC (X ,x)).

Definition 2.3: A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be :

(i, j)–semi-I-continuous [2] if $f^{-1}(V) \in$ (i, j)–SO (X) for every $V \in \boldsymbol{\sigma}_i$.
(i ,j)-irresolute [1] if $f^{-1}(V) \in$ (i, j)-SO (X) for every $\boldsymbol{V} \in$ (i, j)–SO (Y).
Pre-irresolute if $f(V) \in$ (i, j)-SO (Y) for every $\boldsymbol{V} \in$ (i, j)–SO (X).
(i, j)–semi-I-homeomorphism if f is bijective, (i, j)–irresolute and pre-(i, j)–semi-I-open.

Lemma 2.4: If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is an (i, j)–semi-I-homeomorphism then:

(i ,j)–s Cl (f (A)) = f ((i, j)-s Cl (A)) for all $A \subset X$;
(i j)–s Int (f (A)) = f ((i ,j)-s Int (A)) for all $A \subset X$.

Definition 2.5: A subset U of (X, τ_1, τ_2) is called an (i, j)–semi-I-neighbourhood of a point $x \in X$ if there exists an (i, j)–semi-I-open set V of (X, τ_1, τ_2) such that $x \in V \subset U$.

III. QUASI TOPOLOGICAL GROUPS

Definition 3.1: $(G, o, \tau_1, \tau_2, I)$ is (i, j)–semi-I-bitopological group if (G, o) is a group, (G, τ_1, τ_2) is a bitopological space and left translation $L_x : G \rightarrow G$ for all $x \in G$ and right translation $R_x : G \rightarrow G$ for all $x \in G$ are (i, j)–semi-I-continuous and the mapping of inversion $i : G \rightarrow G$ defined by $i(x) = x^{-1}$ is (i ,j)–semi-I-continuous on G.

Theorem 3.2: Let $(G, o, \tau_1, \tau_2, I)$ be an (i, j)–semi- I-bitopological group and $\boldsymbol{\beta}_e$ be the base at identity element e of G. Then:

For every $U \in \boldsymbol{\beta}_e$, there is an element $V \in$ (i, j)–SIO (G, e) such that $V^{-1} \subset U$.
For every $U \in \boldsymbol{\beta}_e$, there is an element $V\ o\ x \subset U$ and $x\ o\ V \subset U$ for each $x \in U$.

Proof: (1) Since $(G, o, \tau_1, \tau_2, I)$ is an *(i, j)*–semi-I-bitopological group. Therefore, for every $U \in \beta_e$ there exists $V \in (i, j)$–*SIO* (G, e) such that $i(V) = v^{-1} \in U$ because the inverse mapping $i : G \to G$ is *(i, j)*–semi-I-continuous.
(2) Since $(G, o, \tau_1, \tau_2, I)$ is and *(i, j)*–semi-I-bitopological group. Thus, for each $U \in \tau_i$ containing x, there exists $V \in (i, j)$–*SIO* (G, e) such that $R_x(V) = V^o x \subset U$.

Lemma 3.3: Let A be a subset of an (i, j)–semi-I-bitopological group (G, o, τ, G). Then (i, j)–sICl $(A^{-1}) \in$ i Cl (A^{-1}).

Proof: Let $x \in ((i, j)\text{–}s\ Cl(A))^{-1}$ and let $U \in \tau_i$ containing x. Then, U^{-1} is an *(i, j)*–semi-I-open neighbourhood of x^{-1}. Since $x^{-1} \in (i, j)\text{–}s\ Cl(A)$, therefore, $U^{-1} \cap A \neq \emptyset$. This implies that $U \cap A^{-1} \neq \emptyset$. That is, $x \in i\ Cl(A^{-1})$ and so $((i, j)\text{–}s\ Cl(A))^{-1} \subset i\ Cl(A^{-1})$.

Remark 3.4: If $(G, o, \tau_1, \tau_2, I)$ is a quasi bitopological group, then τ_i^{-1} is also a topology on G, called conjugate topology of τ_i. If $(G, o, \tau_1, \tau_2, I)$ is a quasi bitopological group, then so is $(G, o, \tau_1^{-1}, \tau_2^{-1}, I)$. Note that $\tau_i^{-1} = \{A \subset G: A^{-1} \in \tau_i\}$. We give the following important result.

Theorem 3.5: Let $(G, o, \tau_1, \tau_2, I)$ be an (i, j)–semi-I-bitopological group. If U is (i, j)–semi-I-open set in $(G, o, \tau_1, \tau_2, I)$, then U^{-1} is (i, j)–semi-I-open in $(G, o, \tau_1^{-1}, \tau_2^{-1}, I)$.

Proof: The proof follows from the respective definitions.

Theorem 3.6: If (G, o, τ_1, τ_2) is (i, j)–semi-I-bitopological group, then $(G, o, \tau_1^{-1}, \tau_2^{-1}, I)$ is also an (i, j)–semi-bitopological group.

Proof: Since (G, o) is a group, (G, τ_1, τ_2) is a bitopological space. Therefore, $(G, o, \tau_1^{-1}, \tau_2^{-1}, I)$ is again a bitopological group. We need to prove that : $i = (G, o, \tau_1^{-1}, \tau_2^{-1}, I) \to (G, o, \tau_1^{-1}, \tau_2^{-1}, I)$, and $L_x : (G, o, \tau_1^{-1}, \tau_2^{-1}, I) \to (G, o, \tau_1^{-1}, \tau_2^{-1}, I)$ and $R_x : (G, \tau^{-1}) \to (G, \tau^{-1})$ are *(i, j)*–semi-I-continuous mappings. First, we show that L_x is *(i, j)*–semi-I-continuous. For this, let $V \in \tau_i^{-1}$. Then $V^{-1} = U \in \tau_i$. Since $(G, o, \tau_1, \tau_2, I)$ is *(i, j)*–semi-I-bitopological group, the left (right) translation is *(i, j)*–semi-I-continuous. Hence $L_x^{\leftarrow}(U) \in (i, j)\text{–}SIO(G, \tau_1, \tau_2)$, that is, $(U\ o\ x^{-1})^{-1} \in (i, j)\text{–}SIO(G, \tau_1^{-1}, \tau_2^{-1})$, that is, $U\ o\ x^{-1} = V^{-1} o\ x^{-1} = (x\ o\ v)^{-1} = L_x^{\leftarrow}(V) \in (i, j)\text{-}SIO(G, \tau_1^{-1}, \tau_2^{-1})$. This proves that $L_x: (G, o, \tau_1^{-1}, \tau_2^{-1}, I) \to (G, o, \tau_1^{-1}, \tau_2^{-1}, I)$ is *(i, j)*–semi-I-continuous for every $x \in G$. Similarly, we can prove that right translation $R_x : (G, o, \tau_1^{-1}, \tau_2^{-1}, I) \to (G, o, \tau_1^{-1}, \tau_2^{-1}, I)$ is *(i, j)*–semi-I-continuous. Trivially $i : (G, o, \tau_1^{-1}, \tau_2^{-1}, I) \to (G, o, \tau_1^{-1}, \tau_2^{-1}, I)$ is continuous and hence *(i, j)*–semi-I-continuous. Hence $(G, o, \tau_1^{-1}, \tau_2^{-1}, I)$ is also an *(i, j)*–semi-I-bitopological group.

Theorem 3.7: If H is a discrete subgroup of an (i, j)–semi-I-bitopological group $(G, o, \tau_1^{-1}, \tau_2^{-1}, I)$, then (i, j)–s Cl (H) is a subgroup of G.

Proof: Let $x, y \in (i, j)\text{–}s\ Cl(H)$. If U and V are respective open neighbourhoods of x and y, then $L_x^{-1}(U) = x^{-1}\ o\ U$ and $L_y^{-1}(U) = y^{-1}\ o\ U$ are *(i, j)*–semi-I-open neighbourhood of e. Since H is a discrete subgroup of an *(i, j)*–semi-I--bitopological group *G*, therefore, $x^{-1} o\ U \cap H \neq \emptyset$ and $y^{-1}\ o\ U \cap H \neq \emptyset$. Therefore, $(x\ o\ y^{-1}\ o\ x^{-1}\ o\ U \cap x\ o\ y^{-1}\ o\ H) \cup (x\ o\ y^{-1}\ o\ y^{-1}\ o\ V \cap x\ o\ y^{-1}\ o\ H) \neq \emptyset$. That is $W \cap x^{-1}\ o\ y^{-1}\ o\ H \neq \emptyset$, where $W = x\ o\ y^{-1}\ o\ x^{-1}\ U \cup x\ o\ y^{-1}\ o\ y^{-1}\ o\ V$ is an *(i, j)*–semi-I-open neighbourhood of $x\ o\ y^{-1}$. Thus, for each $x, y \in (i, j)\text{- } s\ Cl(H)$ implies that $x\ o\ y^{-1} \in (i, j)\text{–}s\ Cl(H)$. Hence $(i, j)\text{–}s\ Cl(H)$ is a subgroup of G.

Corollary 3.8: If H is a discrete subgroup of an (i, j)–semi-I-bitopological group (G, o, τ_1, τ_2) then i Cl (H) is a subgroup of G.

Theorem 3.9: Let $(G, o, \tau_1, \tau_2, I)$ be an (i, j)–semi-I-bitopological group. If A is open in G, then A o B and B o A are (i, j)–semi-I-open in $(G, o, \tau_1, \tau_2, I)$ for any subset B of G.

Proof: Let $x \in B$ and $Z \in A\ o\ x$ we show that z is *(i, j)*–semi-I-interior point of $A\ o\ x$, Let $z = y\ o\ x$ for some $y \in A = A\ o\ x\ o\ x^{-1}$. This implies that $y = z\ o\ x^{-1}$. Now $R_{x\text{-}1}: G \to G$ is *(i ,j)*–semi-I-contiunous, that is, for every open set containing $R_{x\text{-}1}(z) = z\ o\ x^{-1} = y$, there exists an *(i, j)*–semi-I-open set M_z containing z such that $R_{x\text{-}1}(M_z) \subset A$. This implies $M_z\ o\ x^{-1} \subset A$ or $M_z \subset A\ o\ x$. This implies z is *(i, j)*–semi-I-interior point of $A\ o\ x$. Thus $A\ o\ x$ is *(i, j)*–semi-I-open. This implies $A\ o\ B = \cup \bigcup_{x \in B} \boldsymbol{A\ o\ x}$ is *(i, j)*–semi-I-open in (G, o, τ_1, τ_2). Similarly we can prove that for every open set A of G and arbitrary subset B of G, $B\ o\ A$ is *(i ,j)*–semi-I-open in an *(i ,j)*–semi-I-bitopological group $(G, o, \tau_1, \tau_2, I)$.

Definition 3.10: A bijective mapping f : $(X, \tau_X, G) \to (Y, \tau_Y, G)$ ia called quasi s- G -homeomorphism if it is (i ,j)–semi-I-continuous and (i, j)–semi-I-open.

Theorem 3.11: Let $(G, o, \tau_1, \tau_2, I)$ be an (i, j)–semi-I-bitopological group. Then each left (right) translation L_x: G →G is R_x : G →G a quasi s- G -homeomorphism.

Proof: Since $(G, o, \tau_1, \tau_2, I)$ is *(i, j)*–semi-I-bitopological group. Therefore, $L_x: G \to G$ is *(i, j)*–semi-I-continuous. So it is enough to show that $L_x: G \to G$ is *(i, j)*–semi-I-open. Let V be an open set in G. Then by Theroem 3.9, $L_g(V) = g\ o\ V \in SO(G)$. Hence $L_x: G \to G$ is an *(i, j)*–semi-I-open mapping.

Theorem 3.12: Suppose that a subgroup H of an (i, j)–semi-I-bitopological group (G, o, τ_1, τ_2) contains a nonempty open subset of G. Then H is semi-I-open in G.

Proof: By Theorem 3.11 for every $g \in H$, $R_g: G \to G$ is quasi s-homeomorphism. Let U be a nonempty open subset of G with $U \subset H$, then for every $g \in H$, the set $R_g(U) = U\ o\ g$ is *(i, j)*–semi-I-open in (G, o, τ_1, τ_2). Now $H = U\{U\ o\ g : g \in H\}$ is *(i, j)*–semi-I-open in G being union of *(i, j)*–semi-I-open sets of G.

Definition 3.13: A topological space (G, τ) is said to be quasi s-homeogeneous if for all x, y $\in$ G, there is a quasi s-homemorphism f of the space G onto itself such that f (x) = y.

Theorem 3.14: If (G, o, τ_1, τ_2,I) is an (i, j)–semi-I-bitopological group, then every open subgroup of G is also (i, j)–semi-I-closed.

Proof: Since *(G, o, τ_1, τ_2 ,I)* is an *(i, j)*– semi-I-bitopological group and *H* is an open subgroup of *G*. Then any left or right translation *x o H* or *H o x* is *(i, j)*– semi-I-open for each $x \in G$. So *Y= {x o H: $x \in G$}* of all left cosets of *H* in *G* forms a partition of *G*, Thus *Y* is an *(i, j)*–semi-I-open covering of *G*. by disjoint *(i, j)*–semi-I-open sets of *G*. This gives *G\H* is union of *(i, j)*–semi-I-open sets and hence *(i, j)*–semi-I-open. This proves that *H* is *(i, j)*–semi-I-closed.

Corollary 3.15: Every (i , j)–semi-I-bitopological group is a quasi s- homogeneous space.

Proof: Let us take element *x* and *y* in *(G, o, τ_1, τ_2,I)*, and put $z = x^{-1} o\, y$. Since $R_x: G \rightarrow G$ is a quasi s- homeomorphism of *(G, o, τ_1, τ_2,I)* and $R_z (x) = x\, o\, z = x\, o\, (x^{-1} o\, y) = e\, o\, y = y$. Hence *(G, o, τ_1, τ_2)* is quasi s- homogeneous space.

Lemma 3.16: If f: (X, τ_X) $\rightarrow$ (Y, τ_Y) is (i, j)-semi-I-continuous and H is open subset of X, then f_H : (H, τ_H) $\rightarrow$ (Y, τ_Y) is (i, j)–semi-I-continuous.

Theorem 3.17: Every open subgroup H of an (i , j)–semi-I-bitopological group (G, o, τ_1, τ_2) is also an (i , j)–semi-I-bitopological group (called (i , j)–semi-I-bitopological subgroup) of G.

Proof: Let *(G, o, τ_1, τ_2)* be an *(i, j)*–semi-I-bitopological group and *H* be an open subgroup of *G*. We need to prove that *(H, o, τ_H)* is an *(i , j)*–semi-I-bitopological group. For this, We show that $i : H \rightarrow H$, $L_x : H \rightarrow H$ and $R_x : H \rightarrow H$ are *(i , j)*–semi-I-continuous with respect to the relative topology. Since *H* is an open subgroup of *G*. Then by Lemma 3.16, $i_H : (H, \tau_H) \rightarrow (Y, \tau_Y)$, $L_H : (H, \tau_H) \rightarrow (Y, \tau_Y)$ and $R_H :(H, \tau_H) \rightarrow (Y, \tau_Y)$ are *(i , j)*–semi-I-continuous. This proves that *(H, o, τ_H)* is an *(i, j)*–semi-I-bitopological group.

Theorem 3.18: Let f : (G, o, τ_G ,I) $\rightarrow$ (H, o, τ_H ,I) be a homomorphism of (i , j)–semi-I-bitopological groups. If f is (i , j)–irresolute at the natural (identity) element e_G, then f is (i , j)–semi-I-continuous on G.

Proof: Let $x \in G$ be an arbitrary element. Suppose that *W* is an open neighbourhood of $y = f (x) \in H$. Since the left translation in *H* is an *(i , j)*–semi-I-continuous mapping, there is an *(i , j)*–semi-I-open neighbourhood *V* of the nautral element e_H of *H* such that $L_y (V)= y\, o\, V \subset W$. Since *f* is *(i, j)*–irresolute at e_G, therefore, $f (U) \subset V$ for some *(i, j)* –semi-I-open neighbourhood *U* of e_G in *G*. Since $f (U) \subset V$, now $y\, o\, f (U) \subset y\, o\, V \subset W$. This implies $(x\, o\, U) \subset W$. Since *(G, o, τ_G)* is an *(i j)*–semi-I-bitopoogical group, *x o U* is *(i , j)*–semi-I-open in *G*. This proves that *f* is *(i, j)*–semi-I-continuous at *x*. Since *x* was the arbitrary element of *G*, *f* is *(i ,j)*–semi-I-conitinuous on *G*.

REFERENCES:

[1] S. N. Maheshwari and R. Prasad, On pairwise irresolute functions, Mathematical, 18(41) 2(1976), 169-172.

[2] S. N. Maheshwari and R. Prasad, Semiopen sets and semicontinuous functions in bitopological spaces, Math. Notae, 26(1977/78), 29-37.

[3] R. Vaidyanathaswamy, The localisation theory in set topology, Proc. Indian Acad. Sci., 20(1945), 51-61.