Properties of (i,j)-Semi-I-Irresolute Multifunctions

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***Abstract*– In this paper, we define upper (lower) (i,j)-semi- I - irresolute multifunction and obtain some characterizations and basic properties of such a multifunction.**

2000 Mathematics Subject Classification. 54C05, 54C601, 54C08, 54C10.

Key words and phrases– Ideal bitopological space, (i,j)-semi-I-open set, (i,j)-semi-I-irresolute multifunction.

I. INTRODUCTION

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions [2, 8, 9, 10, 11]. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from (i,j)-semiviously existing ones. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [7] and Vaidyanathaswamy, [13]. An ideal I on a topological space (X,(τ_1,τ_2)) is a nonempty collection of subsets of X which satisfies (i) A ∈ I and B ⊂ A implies B ∈ I and (ii) A ∈ I and B ∈ I implies A ∪ B ∈ I . Given a topological space (X,(τ_1,τ_2)) with an ideal I on X and if P(X) is the set of all subsets of X, a set operator (.)* : P(X) → P(X), called the local function [13] of A with respect to τ and I, is defined as follows: for A ⊂ X, A*(τ,I) = {x ∈ X|U ∩ A ∉ I for every U ∈ τ(x)}, where τ(x) = {U ∈ τ|x ∈ U}. A Kuratowski closure operator Cl*(·) for a topology τ*(τ,I) called the ∗-topology, finer than τ is defined by Cl*(A) = A ∪ A*(τ,I) when there is no chance of confusion, A*(I) is denoted by A*. If I is an ideal on X, then (X,τ_1,τ_2,I) is called an ideal bitopological space. In 1996, Dontchev [1] introduced the notion of (i,j)-semi-I-open sets and (i,j)-semi-I-continuous functions. Recently, Akdag [2] introduced and studied the concept of semi-I-continuous multifunction in topological spaces. The purpose of this paper is to define upper (lower) (i,j)-semi-I-irresolute multifunction and to obtain several characterizations of such a multifunction.

II. PRELIMINARIES

Throughout this paper, (X,τ_1,τ_2,I) and (Y,σ_1,σ_2,J) always mean ideal bitopological spaces in which no separation axioms are assumed unless explicitly stated. For a subset A of (X,τ), Cl(A) and Int(A) denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset Sof an ideal topological space (X,τ_1,τ_2,I) is (i,j)-semi-I-open S ⊂ Clj*(Inti(S))). The complement of an (i,j)-semi-I-closed set is said to be an (i,j)-semi-I-open set. The (i,j)-semi-I-closure and the (i,j)-semi-I-interior, that can be defined in the same way as Cl(A) and Int(A), respectively, will be denoted by (i,j)-sICl(A) and (i,j)-sIInt(A), respectively. The family of all (i,j)semi-I-open (resp.(i,j)semi-I-closed) sets of (X,τ_1,τ_2,I) is denoted by (i,j)-SIO(X) (resp. (i,j)-SIC(X)). The family of all (i,j)-semi-I-open (resp. (i,j)-semi-I-closed) sets of (X,τ_1,τ_2,I) containing a point x ∈ X is denoted by (i,j)-SIO(X,x) (resp. (i,j)-SIC(X,x)). A subset U of X is called an (i,j)-semi-I-neighborhood of a point x ∈ X if there exists V ∈ (i,j)-SIO(X,x) such that V ⊂ U. By a multifunction F : (X,τ) → (Y,σ), following [3], we shall denote the upper and lower inverse of a set B of Y by F^+(B) and F^-(B), respectively, that is, F^+(B) = {x ∈ X : F(x) ⊂ B} and F^-(B) = {x ∈ X : F(x) ∩ B ≠ ∅}. In particular, F^-(Y) = {x ∈ X : y∈ F(x)} for each point y ∈ Y and for each A ⊂ X, F(A) = $U_{x\in A}$ F(x). Then F is said to be surjection if F(x) = y.

Lemma 2.1. If A is a subset of (X,τ_1,τ_2,I), then (i,j)-sI Cl(A) = A∪Cli(Intj*(A)).

Definition 2.2. A multifunction F (X,τ_1,τ_2,I) → (Y,σ_1,σ_2,J) is said to be:

(1) upper (i,j)-semi-I-continuous [2] if for each point x ∈ X and each open set V containing F(x), there exists U ∈ (i,j)-SIO(X,x) such that F(U) ⊂ V ;

(2) lower (i,j)-semi-I-continuous [2] if for each point x ∈ X and each open set V such that F(x)∩V ≠ ∅, there exists U ∈ (i,j)-SIO(X,x) such that U ⊂ F^-(V).

(3) On upper and lower (i,j)-semi-I-irresolute multifunctions

Definition 3.1. A multifunction F : (X,τ_1,τ_2,I) → (Y,σ_1,σ_2,J) is said to be:

(1) upper (i,j)semi-I-irresolute if for each point x ∈ X and each (i,j)semi-J-open set V containing F(x), there exists U ∈ (i,j)-SIO(X,x) such that F(U) ⊂ V ;

(2) lower (i,j)-semi-I-irresolute if for each point x ∈ X and each (i,j)-semi-J-open set V such that F(x)∩V ≠ ∅, there exists U ∈ (i,j)-SIO(X,x) such that U ⊂ F^-(V).

Theorem 3.2. The following statements are equivalent for a multifunction F : (X,τ_1,τ_2,I) → (Y,σ_1,σ_2,J):

(1) F is upper (i,j)-semi-I-irresolute;
(2) for each point x of X and each (i,j)semi-J-neighborhood V of F(x), F^+(V) is a (i,j)semi-I-neighborhood of x;
(3) for each point x of X and each (i,j)semi-J-neighborhood V of F(x), there exists a(i,j)semi-I-neighborhood U of x such that F(U) ⊂ V;
(4) F^+(V) ∈ (i,j)-SIO(X) for every V ∈ (i,j)-SJO(Y);
(5) F^-(V) ∈ (i,j)-SIC(X) for every V ∈ (i,j)-SJC(Y);
(6) (i,j)-sICl(F^-(B)) ⊂ F^-((i,j)-sJ Cl(B)) for every subset B of Y .

Proof. (1) ⇒ (2): Let x ∈ X and W be an (i,j)-semi-J-neighborhood of F(x). There exists V ∈ (i,j)-SJO(Y) such that F(x) ⊂ V ⊂ W. Since F is upper (i,j)-semi-I-irresolute, there exists U ∈ (i,j)-SIO(X,x) such that F(U) ⊂ V . Therefore, we have x ∈ U ⊂ F^+(V) ⊂ F^+(W); hence F^+(W) is an (i,j)-semi-I-neighborhood of x.

(2) ⇒ (3): Let x ∈ X and V be an (i,j)-semi-J-neighborhood of F(x). Put U = F^+(V). Then, by (2), U is an (i,j)-semi-I-neighborhood of x and F(U) ⊂ V .

(3) ⇒ (4): Let V ∈ (i,j)-SJO(Y) and x ∈ F^+(V). There exists an (i,j)-semi-I-neighborhood G of x such that F(G) ⊂ V . Therefore, for some U ∈ (i,j)-SIO(X,x) such that U ⊂ G and F(U) ⊂ V . Therefore, we obtain x ∈ U ⊂ F^+(V); hence F^+(V) ∈ (i,j)-SIO(X).

(4) ⇒ (5): Let K be an (i,j)-semi-J-closed set of Y . We have X\F^-(K) = F^+(Y \K) ∈ (i,j)-SJO(X); hence F^-(K) ∈ (i,j)-SIC(X).

(5) ⇒ (6): Let B be any subset of Y. Since (i,j)-sJCl(B) is (i,j)-semi-J-closed in Y , F^-((i,j)-sJ Cl(B)) is (i,j)-semi-I-closed in X and F^-(B) ⊂ F^-((i,j)-sJ Cl(B)). Therefore, we obtain (i,j)-sI Cl(F^-(B)) ⊂ F^-((i,j)-sJ Cl(B)).

(6) ⇒ (1): Let x ∈ X and V ∈ (i,j)-SJO(Y) with F(x) ⊂ V . Then we have F(x)∩(Y \V) = ∅; hence x /∈ F^-(Y \V). By (6), x ∈ (i,j)-sI Cl(F^-(Y \V)) and there exists U ∈ (i,j)-SIO(X,x) such that U ∩ F^-(Y \V) = ∅. Therefore, we obtain F(U) ⊂ V and hence F is upper (i,j)-semi-I-irresolute.

Theorem 3.3. The following statements are equivalent for a multifunction F : (X,τ_1,τ_2,I) → (Y,σ_1,σ_2,J)

(1) F is lower (i,j)-semi-I-irresolute;
(2) For each V ∈ (i,j)-SJO(Y) and each x ∈ F^-(V), there exists U ∈ (i,j)-SIO(X,x) such that U ⊂ F^-(V);
(3) F^-(V) ∈ (i,j)-SIO(X) for every V ∈ (i,j)-SJO(Y);
(4) F^+(K) ∈ (i,j)-SIC(X) for every K ∈ (i,j)-SJC(Y);
(5) F((i,j)-sI Cl(A)) ⊂ (i,j)-sJ Cl(F(A)) for every subset A of X;
(6) (i,j)-sI Cl(F^+(B)) ⊂ F^+((i,j)-sJ Cl(B)) for every subset B of Y .

Proof. (1) ⇒ (2): This is obvious.

(2) ⇒ (3): Let V ∈ (i,j)-SJO(Y) and x ∈ F^-(V). There exists U ∈ (i,j)-SIO(X,x) such that U ⊂ F^-(V). Therefore, we have x ∈ U ⊂ Inti(Clj*(U)) ⊂ Inti(Clj*(F^-(V))); hence F^-(V) ∈ (i,j)-SIO(X).
(3) ⇒ (4): Let K be an (i,j)-semi-I-closed set of Y . We have X\F^+(K) = F^-(Y \K) ∈ (i,j)-SIO(X); hence F^+(K) ∈ (i,j)-SIC(X).
(4) ⇒ (5) and (5) ⇒ (6): Straightforward.
(6) ⇒ (1): Let x ∈ X and V ∈ (i,j)-SJO(Y) with F(x)∩V ≠ ∅. Then F(x) is not a subset of Y \V and x /∈ F^+(Y \V). Since Y \V is (i,j)-semi-I-closed in Y , by (6), x /∈ (i,j)-sI Cl(F^+(Y \V)) and there exists U ∈ (i,j)-SIO(X,x) such that ∅= U ∩ F^-(Y \V) = U ∩ (X\F^-(V)). Therefore, we obtain U ⊂ F^-(V); hence F is lower (i,j)-semi-I-irresolute.

Lemma 3.4. If F : (X,τ_1,τ_2,I) → (Y,σ_1,σ_2,J) is a multifunction, then ((i,j)-sI ClF)$^-$(V) = F^-(V) for each V ∈ (i,j)-SJO(Y).

Proof. Let V ∈ (i,j)-SJO(Y) and x ∈ ((i,j)-sI ClF)$^-$(V). Then V ∩((i,j)-sI ClF)(x) 6= ∅. Since V ∈ (i,j)-SJO(Y), we have V ∩ F(x) ≠ ∅ and hence x ∈ F^-(V). Conversely, let x ∈ F^-(V). Then ∅ ≠ F(x) ∩ V ⊂ ((i,j)-sI ClF)(x) ∩ V and hence x ∈ ((i,j)-sI ClF)$^-$(V). Therefore, we obtain ((i,j)-sI ClF)$^-$(V) = F^-(V).

Theorem 3.5. A multifunction F : (X,τ_1,τ_2,I) → (Y,σ_1,σ_2,J) is lower (i,j)-semi-I-irresolute if and only if (i,j)-sI ClF : (X,τ_1,τ_2,I) → (Y,σ_1,σ_2,J) is lower (i,j)-semi-I-irresolute

Proof. The proof is an immediate consequence of Lemma 3.4 and Theorem 3.3 (iii).

Definition 3.6. A subset A of a topological space (X,τ) is said to be:

(1) α-regular [6] (resp. α-(i,j)-semi-I-regular) if for each a ∈ A and any open (resp. (i,j)-semi-I-open) set U containing a, there exists an open set G of X such that a ∈ G ⊂ Cl(G) ⊂ U;
(2) α-paracompact [6] if every X-open cover A has an X-open refinement which covers A and is locally finite for each point of X.

Lemma 3.7. If A is an α-(i,j)-semi-I-regular, α-paracompact subset of a space X and U is (i,j)-semi-I-neighborhood of A, then there exists an open set G of X such that A ⊂ G ⊂ Cl(G) ⊂ U.

Proof. The proof is similar to that [[6], Theorem 2.5].

Definition 3.8. A multifunction F : $(X,\tau_1,\tau_2,I) \to (Y,\sigma_1,\sigma_2,J)$ is said to be punctually α-paracompact (resp. punctually α-(i,j)-semi-I-regular, punctually α-regular) if for each x ∈ X, F(x) is α-paracompact (resp. α-(i,j)-semi-I-regular, α-regular).

Lemma 3.9. If F : $(X,\tau_1,\tau_2,I) \to (Y,\sigma_1,\sigma_2,J)$ is punctually α-paracompact and punctually α-(i,j)-semi-I-regular, $((i,j)\text{-sI ClF})^+(V) = F^+(V)$ for each V ∈ (i,j)-SJO(Y).

Proof. Let V ∈ (i,j)-SJO(Y). Suppose that x ∈ $((i,j)\text{-sI ClF})^+(V)$. Then, we have F(x) ⊂ (i,j)-sI Cl(F(x)) ⊂ V and hence x ∈ $F^+(V)$. Therefore, we obtain $((i,j)\text{-sI ClF})^+(V) \subset F^+(V)$. Conversely, suppose that x ∈ $F^+(V)$. Then F(x) ⊂ V and by Lemma 3.7 we have F(x) ⊂ G ⊂ Cl(G) ⊂ V for some open set G of Y . Therefore, ((i,j)-sI ClF)(x) ⊂ V and hence x ∈ $((i,j)\text{-sI ClF})^+(V)$. Thus, we obtain $F^+(V) \subset ((i,j)\text{-sIClF})^+(V)$; hence $((i,j)\text{-sI ClF})^+(V) = F^+(V)$.

Theorem 3.10. Let F : $(X,\tau_1,\tau_2,I) \to (Y,\sigma_1,\sigma_2,J)$ be punctually α-paracompact and punctually α-(i,j)-semi-I-regular multifunction. Then F is upper (i,j)-semi-Iirresolute if and only if (i,j)-sI ClF : (X,τ,I) → (Y,σ_1,σ_2,J) is upper (i,j)-semi-Iirresolute.

Proof. The proof follows from Lemma 3.9.

Definition 3.11. An ideal bitopological space (X,τ_1,τ_2,I) is said to be (i,j)-semi-I-normal if for any pair of disjoint closed subsets A, B of X, there exist disjoint U,V ∈ (i,j)-SIO(X) such that A ⊂ U and B ⊂V .

Theorem 3.12. If Y is (i,j)-semi-I-normal and $F_i : X_i \to Y$ is an upper (i,j)-semi-I-irresolute multifunction such that F_i is punctually closed for i = 1,2 and the product of two (i,j)-semi-I-open sets is (i,j)-semi-I-open, then the set $\{(x_1,x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ is (i,j)-semi-I-closed in $X_1 \times X_2$.

Proof. Let $A = \{(x_1,x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ and $(x_1,x_2) \in (X_1 \times X_2)\backslash A$. Then $F_1(x_1) \cap F_2(x_2) = \emptyset$. Since Y is (i,j)-semi-I-normal and F_i is punctually closed for i = 1,2, there exist disjoint V_1,V_2 ∈ (i,j)-SIO(X) such that $F_i(x_i) \subset V_i$ for i = 1,2. Since F_i is upper (i,j)-semi-I-irresolute, $F_i^+(V_i) \in (i,j)\text{-SIO}(X_i,x_i)$ for i = 1,2. Put U= $F^+_1(V_1) \times F^+_2(V_2)$, then U ∈ (i,j)-SIO$(X_1 \times X_2)$ and $(x_1,x_2) \in U \subset (X_1 \times X_2)\backslash A$. This shows that $(X_1 \times X_2)\backslash A \in (i,j)\text{-SIO}(X_1 \times X_2)$; hence A is (i,j)-semi-I-closed set in $X_1 \times X_2$.

Definition 3.13. Let A be a subset of an ideal bitopological space X. The (i,j)-semi-I-frontier of A denoted by (i,j)-sIFr(A), is defined as follows: (i,j)-sIFr(A) = (i,j)-sI Cl(A) ∩ (i,j)-sI Cl(X\A).

Theorem 3.14. The set of a point x of X at which a multifunction F : $(X,\tau_1,\tau_2,I) \to (Y,\sigma_1,\sigma_2,J)$ is not upper (lower) (i,j)-semi-I-irresolute is identical with the union of the (i,j)-semi-I-frontiers of the upper (lower) inverse images of (i,j)-semi-I-open sets containing (meeting) F(x).

Proof. Let x be a point of X at which F is not upper (i,j)-semi-I-irresolute. Then there exists V ∈ (i,j)-SJO(Y) containing F(x) such that $U \cap (X\backslash F^+(V)) \neq \emptyset$ for each U ∈ (i,j)-SIO(X,x). Then x ∈ (i,j)-sI Cl$(X\backslash F^+(V))$. Since x ∈ $F^+(V)$, we have x ∈ (i,j)-sI Cl$(F^+(Y))$ and hence x ∈ (i,j)-sIFr$(F^+(A))$. Conversely, let V ∈ (i,j)-SJO(Y) containing F(x) and x ∈ (i,j)-sIFr$(F^+(V))$. Now, assume that F is upper (i,j)-semi-I-irresolute at x, then there exists U ∈ (i,j)-SIO(X,x) such that F(U) ⊂ V . Therefore, we obtain x ∈ U ⊂ (i,j)-sI Int$(F^+(V)$. This contradicts that x ∈ (i,j)-sIFr$(F^+(V))$. Thus, F is not upper (i,j)-semi-I-irresolute. The proof of the second case is similar.

Lemma 3.15. For a multifunction F : $(X,\tau_1,\tau_2,I) \to (Y,\sigma_1,\sigma_2,J)$, the following holds:

(1) $G^+_F(A \times B) = A \cap F^+(B)$; (2) $G^-_F(A \times B) = A \cap F^-(B)$ for any subset A X and B of Y .

Theorem 3.16. Let F : $(X,\tau_1,\tau_2,I) \to (Y,\sigma_1,\sigma_2,J)$ be a multifunction. If the graph multifunction of F is upper (lower) (i,j)-semi-I-irresolute, then F is upper (lower) (i,j)-semi-I-irresolute.

Proof. Let x ∈ X and V be any (i,j)-semi-J-open subset of Y containing F(x). Since X ×V is an (i,j)-semi-I-open set of X ×Y and $G_F(x) \subset X \times V$, there exists an (i,j)-semi-I-open set U containing x such that $G_F(U) \subset X \times V$. By Lemma 3.19, we have U ⊂ $G^+_F(X \times V) = F^+(V)$ and F(U) ⊂ V . Thus, F is upper (i,j)-semi-I-irresolute. The proof of the lower (i,j)-semi-I-irresolute of F can be done by the similar manner.

Definition 3.17. An ideal topological space (X,τ_1,τ_2,I) is said to (i,j)-semi-I-T_2 if for each pair of distinct points x and y in X, there exist disjoint (i,j)-semi-I-open sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 3.18. If $F : (X,\tau_1,\tau_2,I) \to (Y,\sigma_1,\sigma_2,J)$ is an upper (i,j)-semi-I-irresolute injective multifunction and point closed from a topological space X to a (i,j)-semi-J -normal space Y , then X is a (i,j)-semi-I-T_2 space.

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$ since F is injective. Since Y is (i,j)-semi-J-normal, there exist disjoint (i,j)-semi-J-open sets U and V containing F(x) and F(y), respectively. Thus, there exist disjoint (i,j)-semi-I-open sets $F^+(U)$ and $F^+(V)$ containing x and y, respectively such $G \subset F^+(U)$ and $W \subset F^+(V)$. Therefore, we obtain $G \cap W = \emptyset$; hence X is (i,j)-semi-I-T_2.

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