Properties of δ-b-Irresolute Multifunctions

P. MARAGATHA MEENAKSHI AND P. GOMATHY
Department of Mathematics, Periyar E.V.R College Tiruchirappalli, Tamilnadu
[2]*Department of Mathematics Education, Bonsecours college of Education, Thanjavur.*

***Abstract*– In this paper, we define upper (lower) δ-b-irresolute multifunction and obtain some characterizations and basic properties of such a multifunction.**

2000 Mathematics Subject Classification. 54C05, 54C601, 54C08, 54C10.

Key words and phrases– Topological space, δ-b-open set, δ-b-irresolute multifunction.

I. INTRODUCTION

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. The purpose of this paper is to define upper (lower) δ-b-irresolute multifunction and to obtain several characterizations of such a multifunction.

II. PRELIMINARIES

Throughout this paper, (X, τ) and (Y, σ) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. For a subset A of (X,τ), Cl(A) and Int(A) denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset A of a topological space (X, τ) is δ-b-open A⊂Int(δ-cl(A))Ucl(int(A)). The complement of a δ-b-closed set is said to be a δ-b-open set. The δ-b-closure and the δ-b-interior, that can be defined in the same way as Cl(A) and Int(A), respectively, will be denoted by δ-bcl(A) and δ-bInt(A), respectively. The family of all δ-b-open (resp. δ-b-closed) sets of (X, τ) is denoted by δ-BO(X) (resp. δ-BC(X)). The family of all δ-b-open (resp. δ-b-closed) sets of (X, τ) containing a point x ∈ X is denoted by δ-BO(X,x) (resp. δ-BC(X,x)). A subset U of X is called a δ-b-neighborhood of a point x ∈ X if there exists V ∈ δ-BO(X, x) such that V ⊂ U. By a multifunction F : (X,τ) → (Y,σ), following [3], we shall denote the upper and lower inverse of a set B of Y by $F^{+}(B)$ and $F^{-}(B)$, respectively, that is, $F^{+}(B) = \{x \in X : F(x) \subset B\}$ and $F^{-}(B) = \{x \in X : F(x) \cap B \neq \varnothing\}$. In particular, $F^{-}(Y) = \{x \in X : y \in F(x)\}$ for each point y ∈ Y and for each A ⊂ X, $F(A) = U_{x\in A} F(x)$. Then F is said to be surjection if F(X) = Y.

Definition 2.2. A multifunction F: (X, τ) → (Y, σ) is said to be:

(1) upper δ-b-continuous [2] if for each point x ∈ X and each open set V containing F(x), there exists U ∈ δ-BO(X,x) such that F(U) ⊂ V ;

(2) lower δ-b-continuous [2] if for each point x ∈ X and each open set V such that $F(x)\cap V \neq \varnothing$, there exists U ∈ δ-BO(X,x) such that $U \subset F^{-}(V)$.

3. On upper and lower δ-b-irresolute multifunctions

Definition 3.1. A multifunction F: (X, τ) → (Y, σ) is said to be:

(1) upper δ-b-irresolute if for each point x ∈ X and each δ-b-open set V containing F(x), there exists U ∈ δ-BO(X,x) such that F(U) ⊂ V ;

(2) lower δ-b-irresolute if for each point x ∈ X and each δ-b-open set V such that $F(x)\cap V \neq \varnothing$, there exists U ∈ δ-BO(X,x) such that $U \subset F^{-}(V)$.

Theorem 3.2. The following statements are equivalent for a multifunction F: (X, τ) → (Y, σ):

(1) F is upper δ-b-irresolute;
(2) for each point x of X and each δ-b-neighborhood V of F(x), $F^{+}(V)$ is a δ-b-neighborhood of x;
(3) for each point x of X and each δ-b-neighborhood V of F(x), there exists a δ-b-neighborhood U of x such that F(U) ⊂ V;
(4) $F^{+}(V) \in$ δ-BO(X) for every V ∈ δ-BO(Y);
(5) $F^{-}(V) \in$ δ-BC(X) for every V ∈ δ-BC(Y);
(6) δ-bCl($F^{-}(B)$) ⊂ F^{-}(δ-bCl(B)) for every subset B of Y .

Proof. (1) ⇒ (2): Let x ∈ X and W be a δ-b-neighborhood of F(x). There exists V ∈ δ-BO(Y) such that F(x) ⊂ V ⊂ W. Since F is upper δ-b-irresolute, there exists U ∈ δ-BO(X,x) such that F(U) ⊂ V. Therefore, we have x ∈ U ⊂ F^+(V) ⊂ F^+(W); hence F^+(W) is a δ-b-neighborhood of x.

(2) ⇒ (3): Let x ∈ X and V be a δ-b-neighborhood of F(x). Put U = F^+(V). Then, by (2), U is a δ-b-neighborhood of x and F(U) ⊂ V .

(3) ⇒ (4): Let V ∈ δ-BO(Y) and x ∈ F^+(V). There exists a δ-b-neighborhood G of x such that F(G) ⊂ V . Therefore, for some U ∈ δ-BO(X,x) such that U ⊂ G and F(U) ⊂ V . Therefore, we obtain x ∈ U ⊂ F^+(V); hence F^+(V) ∈ δ-BO(X).

(4) ⇒ (5): Let K be a δ-b-closed set of Y. We have X\F^-(K) = F^+(Y \K) ∈ δ-BO(X); hence F^-(K) ∈ δ-BC(X).

(5) ⇒ (6): Let B be any subset of Y. Since δ-bcl(B) is δ-b-closed in Y , F^-(δ-bcl(B)) is δ-b-closed in X and F^-(B) ⊂ F^-(δ-bcl (B)). Therefore, we obtain δ-bcl(F^-(B)) ⊂ F^-(δ-bcl(B)).

(6) ⇒ (1): Let x ∈ X and V ∈ δ-BO(Y) with F(x) ⊂ V . Then we have F(x)∩(Y \V) = ∅; hence x /∈ F^-(Y \V). By (6), x ∈ δ-bcl (F^-(Y \V)) and there exists U ∈ δ-BO(X,x) such that U ∩ F^-(Y \V) = ∅. Therefore, we obtain F(U) ⊂ V and hence F is upper b-irresolute.

Theorem 3.3. The following statements are equivalent for a multifunction F: (X, τ) → (Y, σ)

(1) F is lower δ-b-irresolute;
(2) For each V ∈ δ-BO(Y) and each x ∈ F^-(V), there exists U ∈ δ-BO(X,x) such that U ⊂ F^-(V);
(3) F^-(V) ∈ δ-BO(X) for every V ∈ δ-BO(Y);
(4) F^+(K) ∈ δ-BC(X) for every K ∈ (i,j)-SJC(Y);
(5) F(δ-bcl(A)) ⊂ δ-bcl(F(A)) for every subset A of X;
(6) δ-bcl(F^+(B)) ⊂ F^+(δ-bcl(B)) for every subset B of Y .

Proof. (1) ⇒ (2): This is obvious.

(2) ⇒ (3): Let V ∈ δ-BO(Y) and x ∈ F^-(V). There exists U ∈ δ-BO(X,x) such that U ⊂ F^-(V). Therefore, we have x ∈ U ⊂ Int(Cl(U)) ⊂ Int(Cl(F^-(V))); hence F^-(V) ∈ δ-BO(X).
(2) ⇒ (4): Let K be an δ-b-closed set of Y . We have X\F^+(K) = F^-(Y \K) ∈ δ-BO(X); hence F^+(K) ∈ δ-BC(X).
(3) ⇒ (5) and (5) ⇒ (6): Straightforward.
(6) ⇒ (1): Let x ∈ X and V ∈ δ-BO(Y) with F(x)∩V ≠ ∅. Then F(x) is not a subset of Y \V and x /∈ F^+(Y \V). Since Y \V is δ-b-closed in Y, by (6), x /∈ δ-bcl(F^+(Y \V)) and there exists U ∈ δ-BO(X,x) such that ∅= U ∩ F^-(Y \V) = U ∩ (X\F^-(V)). Therefore, we obtain U ⊂ F^-(V); hence F is lower δ-b-irresolute.

Lemma 3.4. If F: (X, τ) → (Y, σ) is a multifunction, then $(δ\text{-bcl}F)^-(V) = F^-(V)$ for each V ∈ δ-BO(Y).

Proof. Let V ∈ δ-BO(Y) and x ∈ $(δ\text{-bcl}F)^-(V)$. Then V ∩ (δ-bclF)(x) ≠ ∅. Since V ∈ δ-BO(Y), we have V ∩ F(x) ≠ ∅ and hence x ∈ $F^-(V)$. Conversely, let x ∈ $F^-(V)$. Then ∅ ≠ F(x) ∩ V ⊂ (δ-bclF)(x) ∩ V and hence x ∈ $(\text{bcl}F)^-(V)$. Therefore, we obtain $(δ\text{-bcl}F)^-(V) = F^-(V)$.

Theorem 3.5. A multifunction F: (X, τ) → (Y, σ) is lower δ-b-irresolute if and only if δ-bclF : (X,τ_1,τ_2,I) → (Y,σ_1, σ_2,J) is lower δ-b-irresolute

Proof. The proof is an immediate consequence of Lemma 3.4 and Theorem 3.3 (iii).

Definition 3.6. A subset A of a topological space (X,τ) is said to be:

(1) α-regular [6] (resp. α-δ-b-regular) if for each a ∈ A and any open (resp. δ-b-open) set U containing a, there exists an open set G of X such that a ∈ G ⊂ Cl(G) ⊂ U;
(2) α-paracompact [6] if every X-open cover A has an X-open refinement which covers A and is locally finite for each point of X.

Lemma 3.7. If A is an α-b-regular, α-paracompact subset of a space X and U is δ-b-neighborhood of A, then there exists an open set G of X such that A ⊂ G ⊂ Cl(G) ⊂ U.

Proof. The proof is similar to that [[6], Theorem 2.5].

Definition 3.8. A multifunction F: (X, τ) → (Y, σ) is said to be punctually α-paracompact (resp. punctually α-b-regular, punctually α-regular) if for each x ∈ X, F(x) is α-paracompact (resp. α-b-regular, α-regular).

Lemma 3.9. If F: (X, τ) → (Y, σ) is punctually α-paracompact and punctually α-δ-b-regular, $(δ\text{-bcl}F)^+(V) = F^+(V)$ for each V ∈ δ-BO(Y).

Proof. Let V ∈ δ-BO(Y). Suppose that x ∈ $(δ\text{-bcl}F)^+(V)$. Then, we have F(x) ⊂ δ-bcl(F(x)) ⊂ V and hence x ∈ $F^+(V)$. Therefore, we obtain $(δ\text{-bcl}F)^+(V) ⊂ F^+(V)$. Conversely, suppose that x ∈ $F^+(V)$. Then F(x) ⊂ V and by Lemma 3.7 we have $F(x) \subset G \subset Cl(G) \subset V$ for some open set G of Y . Therefore, $(\delta\text{-bcl}F)(x) \subset V$ and hence $x \in (\delta\text{-bcl}F)^+(V)$. Thus, we obtain $F^+(V) \subset (\delta\text{-bCl}F)^+(V)$; hence $(\delta\text{-bcl}F)^+(V) = F^+(V)$.

Theorem 3.10. Let F: $(X, \tau) \to (Y, \sigma)$ be punctually α-paracompact and punctually α-δ-b-regular multifunction. Then F is upper δ-b-irresolute if and only if δ-bclF : $(X,\tau) \to (Y, \sigma)$ is upper δ-b-irresolute.

Proof. The proof follows from Lemma 3.9.

Definition 3.11. A topological space (X, τ) is said to be δ-b-normal if for any pair of disjoint closed subsets A, B of X, there exist disjoint $U,V \in \delta\text{-BO}(X)$ such that $A \subset U$ and $B \subset V$.

Theorem 3.12. If Y is δ-b-normal and $F_i : X_i \to Y$ is an upper δ-b-irresolute multifunction such that F_i is punctually closed for i = 1,2 and the product of two δ-b-open sets is δ-b-open, then the set $\{(x_1,x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ is δ-b-closed in $X_1 \times X_2$.

Proof. Let $A = \{(x_1,x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ and $(x_1,x_2) \in (X_1 \times X_2)\backslash A$. Then $F_1(x_1) \cap F_2(x_2) = \emptyset$. Since Y is δ-b-normal and F_i is punctually closed for i = 1,2, there exist disjoint $V_1,V_2 \in \delta\text{-BO}(X)$ such that $F_i(x_i) \subset V_i$ for i = 1,2. Since F_i is upper δ-b-irresolute, $F_i^+(V_i) \in \delta\text{-BO}(X_i,x_i)$ for i = 1,2. Put $U= F^+_1(V_1) \times F^+_2(V_2)$, then $U \in \delta\text{-BO}(X_1 \times X_2)$ and $(x_1,x_2) \in U \subset (X_1 \times X_2)\backslash A$. This shows that $(X_1 \times X_2)\backslash A \in \delta\text{-BO}(X_1 \times X_2)$; hence A is δ-b-closed set in $X_1 \times X_2$.

Definition 3.13. Let A be a subset of a topological space X. The δ-b-frontier of A denoted by δ-b-Fr(A), is defined as follows: $\delta\text{-bFr}(A) = \delta\text{-bcl}(A) \cap \delta\text{-bcl}(X\backslash A)$.

Theorem 3.14. The set of a point x of X at which a multifunction F: $(X, \tau) \to (Y, \sigma)$ is not upper (lower) δ-b-irresolute is identical with the union of the δ-b-frontiers of the upper (lower) inverse images of δ-b-open sets containing (meeting) F(x).

Proof. Let x be a point of X at which F is not upper δ-b-irresolute. Then there exists $V \in \delta\text{-BO}(Y)$ containing F(x) such that $U \cap (X\backslash F^+(V)) \neq \emptyset$ for each $U \in \delta\text{-BO}(X,x)$. Then $x \in \delta\text{-bcl}(X\backslash F^+(V))$. Since $x \in F^+(V)$, we have $x \in \delta\text{-bcl}(F^+(Y)$ and hence $x \in \delta\text{-bFr}(F^+(A))$. Conversely, let $V \in \delta\text{-BO}(Y)$ containing F(x) and $x \in \delta\text{-bFr}(F^+(V))$. Now, assume that F is upper δ-b-irresolute at x, then there exists $U \in \delta\text{-BO}(X,x)$ such that $F(U) \subset V$. Therefore, we obtain $x \in U \subset \delta\text{-bInt}(F^+(V)$. This contradicts that $x \in \delta\text{-bFr}(F^+(V))$. Thus, F is not upper δ-b-irresolute. The proof of the second case is similar.

Lemma 3.15. For a multifunction F: $(X, \tau) \to (Y, \sigma)$, the following holds:

(1) $G^+_F(A \times B) = A \cap F^+(B)$; (2) $G^-_F(A \times B) = A \cap F^-(B)$ for any subset A X and B of Y .

Theorem 3.16. Let F: $(X, \tau) \to (Y, \sigma)$ be a multifunction. If the graph multifunction of F is upper (lower) δ-b-irresolute, then F is upper (lower) δ-b-irresolute.

Proof. Let $x \in X$ and V be any δ-b-open subset of Y containing F(x). Since $X \times V$ is a δ-b-open set of $X \times Y$ and $G_F(x) \subset X \times V$, there exists a δ-b-open set U containing x such that $G_F(U) \subset X\times V$. By Lemma 3.19, we have $U \subset G^+_F(X \times V) = F^+(V)$ and $F(U) \subset V$. Thus, F is upper δ-b-irresolute. The proof of the lower δ-b-irresolute of F can be done by the similar manner.

Definition 3.17. A topological space (X, τ) is said to δ-b-T_2 if for each pair of distinct points x and y in X, there exist disjoint δ-b-open sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 3.18. If F: $(X, \tau) \to (Y, \sigma)$ is an upper δ-b-irresolute injective multifunction and point closed from a topological space X to a δ-b-normal space Y , then X is a δ-b-T_2 space.

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$ since F is injective. Since Y is δ-b-normal, there exist disjoint δ-b-open sets U and V containing F(x) and F(y), respectively. Thus, there exist disjoint δ-b-open sets $F^+(U)$ and $F^+(V)$ containing x and y, respectively such $G \subset F^+(U)$ and $W \subset F^+(V)$. Therefore, we obtain $G \cap W = \emptyset$; hence X is δ-b-T_2.

REFERENCES

[1] M.E. Abd El-Monsef and A. A. Nasef, On Multifunctions, Chaos, Solitons and Fractals, 12(2001), 2387-2394.

[2] D. Andrijevic, Semi-preopen sets, Mat. Vesnik, 38(1986), 24-32.

[3] A. A. El-Atik, A study of some types of mappings on topological spaces, Master's Thesis, Tanta University, Tanta (Egypt), 1995.

[4] I. Kovacevic, Subsets and paracompactness, Univ. u. Novom Sadu, Zb. Rad. Prirod. Mat. Fac. Ser. Mat., 14(1984), 79-87.

[5] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70(1963), 36-41.

[6] O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15(1965), 961-970.

[7]. M. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., 41(1937), 374-381.