β^*g-normal spaces in topological spaces

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Abstract. The aim of this paper is to introduce and study a new class of spaces, called β^*g-normal spaces. The relationships among s-normal spaces, p-normal spaces, α-normal spaces, β-normal spaces, γ-normal spaces and β^*g-normal spaces are investigated. Moreover, we introduce the forms of generalized β^*g-closed and β^*g-generalized closed functions. We obtain characterizations of β^*g-normal spaces, properties of the forms of generalized β^*g-closed functions and preservation theorems.

Key Words: β-open, β^*g-closed, gβ^*g-closed and β^*gg-closed sets; β^*g-normal spaces; β^*g-closed and β^*g-gβ^*g-closed functions.

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1. Introduction

Normality is an important topological property and hence it is of significance both from intrinsic interest as well as from applications view point to obtain factorizations of normality in terms of weaker topological properties. In 1937, Stone [**32**] introduced the concept of regular-open sets. In 1963, Levine [**16**] introduced the notion of semi-open sets and obtained their properties. In 1965, Njastad [**24**] introduced the notion of α-open sets. In 1970, Levine [**17**] initiated the investigation of g-closed sets in topological spaces, since then many modifications of g-closed sets were defined and investigated by a large number of topologists. In 1978, Maheshwari [**18**] introduced the notion of s-normal spaces and obtained their characterizations. In 1982, Mashhour [**23**] introduced the notion of pre-open sets. In 1983, Abd El-Monsef [**1**] introduced the notion of β-open sets. In 1989, Nour [**28**] introduced the notion of pre-normal spaces and obtained their characterizations. In 1990, Arya and Nour [**5**] introduced the concepts of gs-closed sets. In 1990, Mahmoud [**19**] introduced the notion of β-normal spaces and obtained their characterizations. In 1995, Paul [**29**] further investigated p-normal spaces and obtained more characterizations. In 1996, Maki and et. al [**20**] introduced the concepts of gp-closed sets. In 1997, El-Atik [**13**] introduced the notion of γ-open sets. In 2002, by using gp-closed sets, Park [**30**] obtained some characterizations of p-normal spaces and defined pre gp-continuous functions. In 2007, Ekici [**14**] introduced a new class of normal spaces, namely γ-normal spaces, which is a generalizing of the classes of p-normal spaces and s-normal spaces. The relations among γ-normal, p-normal spaces and s-normal spaces and also properties of γ-normal spaces are investigated. In 2009, Benchalli [**8**] introduced the notion of α-normal spaces and obtained their characterizations.

2. Preliminaries

In what follows, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated and $f : (X, \Im) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f of a space $(X, \Im)$ into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by cl(A) and int(A), respectively.

2.1 Definition. A subset A of a space X is said to be:

(1) **regular open** [**32**] if A = int(cl(A)).

(2) **semi-open** [**16**] if $A \subset$ cl(int(A)).

(3) **pre-open** [**23**] or nearly open [13] if $A \subset$ int(cl(A)).

(4) **α-open** [**24**] if $A \subset$ int(cl(int(A))).

(5) **β-open** [**1**] if $A \subset cl(int(cl(A)))$.

(6) **b-open** [**3**] or **γ-open** [**13**] if $A \subset cl(int(A)) \cup int(cl(A))$.

The complement of an α-open (resp. β-open, γ-open, pre-open, semi-open, regular open) set is called **α-closed** (resp. **β-closed**, **γ-closed**, **pre-closed**, **semi-closed**, **regular closed**).

The intersection of all α-closed (resp. β-closed, γ-closed, semi-closed, pre-closed) sets containing A is called the **α-closure** (resp. **β-closure**, **γ-closure**, **semi-closure**, **pre-closure**) of A and is denoted by **α-cl(A)** (resp. **β-cl(A)**, **γ-cl(A)**, **s-cl(A)**, **p-cl(A)**). The **α-interior** (resp. **β-interior**, **γ-interior**, **semi-interior**, **pre-interior**) of A, denoted by **α-int(A)** (resp. **β-int(A)**, **γ-int(A)**, **s-int(A)**, **p-int(A)**) is defined to be the union of all α-open (resp. β-open, γ-open, semi-open, pre-open) sets contained in A.

2.2 Definition. A subset A of a space $(X, \Im)$ is said to be **β^*g-closed** [**31**] if $cl(A) \subset U$ whenever $A \subset U$ and U is β-open in X. The complement of β^*g-closed set is said to be **β^*g-open**.

The intersection of all β^*g-closed sets containing A, is called the **β^*g-closure** of A and is denoted by **β^*g-cl(A)**. The **β^*g-interior** of A, denoted by **β^*g-int(A)** is defined to be the union of all β^*g-open sets contained in A.

The family of all β^*g-open (resp. β^*g-closed, regular open, regular closed, semi-open, semi-closed, pre-open, pre-closed, α-open, α-closed, β-open, β-closed, γ-open, γ-closed) sets of a space X is denoted by $\beta^*gO(X)$ (resp. $\beta^*gC(X)$, RO(X), RC(X), SO(X), SC(X), PO(X), PC(X), αO(X), αC(X), βO(X), βC(X), γO(X), γC(X)).

2.3 Definition. A subset A of a space $(X, \Im)$ is said to be

(1) **g-closed** [**17**], if $cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.

(2) **generalized β^*g-closed** (briefly **gβ^*g-closed**) if $\beta^*g\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.

(3) **β^*g-generalized-closed** (briefly **β^*gg-closed**) if $\beta^*g\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \beta^*gO(X)$.

(4) **gs-closed** [**5**], if $s\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.

(5) **sg-closed** [**6**] if $s\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in SO(X)$.

(6) **gp-closed** [**20**] if $p\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.

(7) **pg-closed** [**7**] if $p\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in PO(X)$.

(8) **αg-closed** [**21**] if $\alpha\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.

(9) **gα-closed** [**21**] if $\alpha\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \alpha O(X)$.

(10) **gβ-closed** [**11**] if $\beta\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.

(11) **βg-closed** [**31**] if $\beta\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \beta O(X)$.

(12) **gγ-closed** [**14**] if $\gamma\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.

(13) **γg-closed** [**14**] if $\gamma\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \gamma O(X)$.

The complement of β^*g-closed (resp. gβ^*g-closed, β^*gg-closed, g-closed, αg-closed, gα-closed, gβ-closed, βg-closed, gγ-closed, γg-closed, gp-closed, pg-closed gs-closed, sg-closed) set is said to be **β^*g-open** (resp. **gβ^*g-open**, **β^*gg-open**, **g-open**, **αg-open**, **gα-open**, **gβ-open**, **βg-open**, **gγ-open**, **γg-open**, **gp-open**, **pg-open**, **gs-open**, **sg-open**).

2.4 Remark. We have the following implications for the properties of subsets:

closed → β^*g-closed → s_*g-closed → g-closed

↓ ↓ ↓

α-closed	→	gα-closed	→	αg-closed
↓				↓
p-closed	→	pg-closed	→	gp-closed
↓				↓
γ-closed	→	γg-closed	→	gγ-closed
↓				↓
β-closed	→	βg-closed	→	gβ-closed

and

closed	→	β^*g-closed	→	s^*g-closed	→	g-closed
		↓		↓		↓
		α-closed	→	gα-closed	→	αg-closed
		↓				↓
		s-closed	→	sg-closed	→	gs-closed
		↓				↓
		γ-closed	→	γg-closed	→	gγ-closed
		↓				↓
		β-closed	→	βg-closed	→	gβ-closed

Where none of the implications is reversible as can be seen from the following examples:

2.5 Example. Let X = {a, b, c, d} and ℑ = {ϕ, {b, d}, {a, b, d}, {b, c, d}, X}. Then

(1) Closed sets in (X, ℑ) are ϕ, X, {a}, {c}, {a, c}.

(2) g-closed sets in (X, ℑ) are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, c, d}.

(3) β^*g-closed sets in (X, ℑ) are ϕ, X, {a}, {c}, {a, c}.

(4) s^*g-closed sets in (X, ℑ) are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, c, d}.

(5) p-closed sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d}.

(6) gp-closed sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d}.

(7) pg-closed sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d}.

(8) s-closed sets in (X, ℑ) are ϕ, X, {a}, {c}, {a, c}.

(9) gs-closed sets in (X, ℑ) are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, c, d}.

(10) sg-closed sets in (X, ℑ) are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, c, d}.

(11) α-closed sets in (X, ℑ) are ϕ, X, {a}, {c}, {a, c}.

(12) αg-closed sets in (X, $\Im$) are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, c, d}.

(13) gα-closed sets in (X, $\Im$) are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, b, d}, {a, c, d}.

(14) β-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d}.

(15) gβ-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d}.

(16) βg-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d}.

(17) γ-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d}.

(18) gγ-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}.

(19) γg-closed sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}.

2.6 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {b}, {a, b}, {a, c}, {a, b, c}, X}. Then

(1) closed sets in (X, $\Im$) are ϕ, X, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(2) g-closed sets in (X, $\Im$) are ϕ, X, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

(3) β^*g-closed sets in (X, $\Im$) are ϕ, X, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(4) s^*g-closed sets in (X, $\Im$) are ϕ, X, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(5) p-closed sets in (X, $\Im$) are ϕ, X, {c}, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(6) gp-closed sets in (X, $\Im$) are ϕ, X, {c}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

(7) pg-closed sets in (X, $\Im$) are ϕ, X, {c}, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(8) s-closed sets in (X, $\Im$) are ϕ, X, {b}, {c}, {d}, {a, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(9) gs-closed sets in (X, $\Im$) are ϕ, X, {b}, {c}, {d}, {a, c}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

(10) sg-closed sets in (X, $\Im$) are ϕ, X, {b}, {c}, {d}, {a, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(11) α-closed sets in (X, $\Im$) are ϕ, X, {c}, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(12) αg-closed sets in (X, $\Im$) are ϕ, X, {c}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

(13) gα-closed sets in (X, $\Im$) are ϕ, X, {c}, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(14) β-closed sets in (X, $\Im$) are ϕ, X, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(15) gβ-closed sets in (X, $\Im$) are ϕ, X,{b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

(16) βg-closed sets in (X, $\Im$) are ϕ, X, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(17) γ-closed sets in (X, $\Im$) are ϕ, X, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(18) gγ-closed sets in (X, $\Im$) are ϕ, X, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.

(19) γg-closed sets in (X, $\Im$) are ϕ, X, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

3. β^*g-normal spaces

3.1 Definition. A space X is said to be **β^*g-normal** if for any pair of disjoint closed sets A and B, there exist disjoint β^*g-open sets U and V such that $A \subset U$ and $B \subset V$.

3.2 Definition. A space X is said to be **pre-normal** [**28**] or **p-normal** [**29**] (resp. **s-normal** [**18**], **α-normal** [**8**], **β-normal** [**19**], **γ-normal** [**14**]) if for any pair of disjoint closed sets A and B, there exist disjoint pre-open (resp. semi-open, α-open, β-open, γ-open) sets U and V such that $A \subset U$ and $B \subset V$.

3.3 Remark. The following diagram holds for a topological space (X, $\mathfrak{I}$):

normal → β^*g-normal → α-normal → s-normal

↓

α-normal ↓

↓

p-normal → γ-normal

↓ ↓

γ-normal → β-normal

None of these implications is reversible as shown by the following examples.

3.4 Example. Let X = {a, b, c, d} and $\mathfrak{I}$ = {∅, {b, d}, {a, b, d}, {b, c, d}, X}. Then the space (X, $\mathfrak{I}$) is γ-normal as well as β-normal. But it is neither α-normal nor s-normal.

3.5 Example. Let X = {a, b, c, d} and $\mathfrak{I}$ = {∅, {b}, {d}, {b, d}, {a, b, d}, {b, c, d}, X}. Then the space (X, $\mathfrak{I}$) is β-normal but not p-normal.

3.6 Example. Let X = {a, b, c, d} and $\mathfrak{I}$ = {∅, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, X}. Then the space (X, $\mathfrak{I}$) is γ-normal as well as β-normal but not p-normal.

3.7 Example. Let X = {a, b, c, d} and $\mathfrak{I}$ = {∅, {a}, {b, c}, X}. Then the space (X, $\mathfrak{I}$) is normal as well as β^*g-normal.

3.8 Theorem. For a space X the following are equivalent:

(1) X is β^*g-normal,

(2) For every pair of open sets U and V whose union is X, there exist β^*g-closed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$,

(3) For every closed set H and every open set K containing H, there exists a β^*g-open set U such that $H \subset U \subset \beta^*g\text{-cl}(U) \subset K$.

Proof: (1) ⇒ (2) : Let U and V be a pair of open sets in a β^*g-normal space X such that $X = U \cup V$. Then $X - U$, $X - V$ are disjoint closed sets. Since X is β^*g-normal, there exist disjoint β^*g-open sets U_1 and V_1 such that $X - U \subset U_1$ and $X - V \subset V_1$. Let $A = X - U_1$, $B = X - V_1$. Then A and B are β^*g-closed sets such that $A \subset U$, $B \subset V$ and $A \cup B = X$.

(2) ⇒ (3) : Let H be a closed set and K be an open set containing H. Then $X - H$ and K are open sets whose union is X. Then by (2), there exist β^*g-closed sets M_1 and M_2 such that $M_1 \subset X - H$ and $M_2 \subset K$ and $M_1 \cup M_2 = X$. Then $H \subset X - M_1$, $X - K \subset X - M_2$ and $(X - M_1) \cap (X - M_2) = \emptyset$. Let $U = X - M_1$ and $V = X - M_2$. Then U and V are disjoint β^*g-open sets such that $H \subset U \subset X - V \subset K$. As $X - V$ is β^*g-closed set, we have $\beta^*g\text{-cl}(U) \subset X - V$ and $H \subset U \subset \beta^*g\text{-cl}(U) \subset K$.

(3) ⇒ (1) : Let H_1 and H_2 be any two disjoint closed sets of X. Put $K = X - H_2$, then $H_2 \cap K = \emptyset$. $H_1 \subset K$, where K is an open set. Then by (3), there exists a β^*g-open set U of X such that $H_1 \subset U \subset \beta^*g\text{-cl}(U) \subset K$. It follows that $H_2 \subset X - \beta^*$g-

cl(U) = V , say, then V is β^*g-open and U ∩ V = ∅. Hence H_1 and H_2 are separated by β^*g-open sets U and V. Therefore X is β^*g-normal.

4. Some related functions with β^*g-normal spaces

4.1 Definition . A function f : X → Y is called

(1) **R-map** [**9**] if $f^{-1}(V)$ is regular open in X for every regular open set V of Y ,

(2) **completely continuous** [**4**] if $f^{-1}(V)$ is regular open in X for every open set V of Y ,

(3) **rc-continuous** [**15**] if for each regular closed set F in Y, $f^{-1}(F)$ is regular closed in X.

4.2 Definition . A function f : X → Y is called

(1) **strongly β^*g-open** if f(U) ∈ β^*gO(Y) for each U ∈ β^*gO(X),

(2) **strongly β^*g-closed** if f(U) ∈ β^*gC(Y) for each U ∈ β^*gC(X),

(3) **almost β^*g-irresolute** if for each x in X and each β^*g-neighbourhood V of f(x), β^*g-cl($f^{-1}(V)$) is a β^*g-neighbourhood of x.

4.3 Theorem . A function f : X → Y is strongly β^*g-closed if and only if for each subset A in Y and for each β^*g-open set U in X containing $f^{-1}(A)$, there exists a β^*g-open set V containing A such that $f^{-1}(V) \subset U$.

Proof: (⇒) : Suppose that f is strongly β^*g-closed. Let A be a subset of Y and U ∈ β^*gO(X) containing $f^{-1}(A)$. Put V = Y − f(X − U), then V is a β^*g-open set of Y such that A ⊂ V and $f^{-1}(V) \subset U$.

(⇐) : Let K be any β^*g-closed set of X. Then $f^{-1}(Y - f(K)) \subset X - K$ and X − K ∈ β^*g O(X). There exists a β^*g-open set V of Y such that Y − f(K) ⊂ V and $f^{-1}(V) \subset X - K$. Therefore, we have f(K) ⊃ Y − V and K ⊂ $f^{-1}(Y - V)$. Hence, we obtain f(K) = Y − V and f(K) is β^*g-closed in Y . This shows that f is strongly β^*g-closed.

4.4 Lemma. For a function f : X → Y , the following are equivalent:

(1) f is almost β^*g-irresolute,

(2) $f^{-1}(V) \subset \beta^*$g-int($\beta$*g-cl($f^{-1}(V)$)) for every V ∈ β^*gO(Y).

4.5 Theorem. A function f : X → Y is almost β^*g-irresolute if and only if f(β^*g-cl(U)) ⊂ β^*g-cl(f(U)) for every U ∈ β^*gO(X).

Proof: (⇒) : Let U ∈ β^*gO(X). Suppose y ∉ β^*g-cl(f(U)). Then there exists V ∈ β^*gO(Y) such that V ∩ f(U) = ∅. Hence, $f^{-1}(V) \cap U = \emptyset$. Since U ∈ β^*gO(X), we have β^*g-int(β^*g-cl($f^{-1}(V)$)) ∩ β^*g-cl(U) = ∅. Then by **Lemma 4.4**, $f^{-1}(V)$ ∩ β^*g-cl(U) = ∅ and hence V ∩ f(β^*g-cl(U)) = ∅. This implies that y ∉ f(β^*g-cl(U)).

(⇐) : If V ∈ β^*gO(Y), then M = X − β^*g-cl($f^{-1}(V)$) ∈ β^*gO(X). By hypothesis, f(β^*g-cl(M)) ⊂ β^*g-cl(f(M)) and hence X − β^*g-int(β^*g-cl($f^{-1}(V)$)) = β^*g-cl(M) ⊂ f^{-1}(β^*g-cl(f(M))) ⊂ f^{-1}(β^*g-cl(f(X − $f^{-1}(V)$))) ⊂ f^{-1}(β^*g-cl(Y \ V)) = $f^{-1}(Y - V) = X - f^{-1}(V)$. Therefore, $f^{-1}(V) \subset \beta^*$g-int($\beta^*$g-cl($f^{-1}(V)$)). By **Lemma 4.4**, f is almost β^*g-irresolute.

4.6 Theorem. If f : X → Y is a strongly β^*g-open continuous almost β^*g-irresolute function from a β^*g-normal space X onto a space Y , then Y is β^*g-normal.

Proof: Let A be a closed subset of Y and B be an open set containing A. Then by continuity of f, $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of X such that $f^{-1}(A) \subset f^{-1}(B)$. As X is β^*g-normal, there exists a β^*g-open set U in X such that $f^{-1}(A) \subset U \subset \beta^*$g-cl(U) ⊂ $f^{-1}(B)$ by **Theorem 3.8**. Then, f($f^{-1}(A)$) ⊂ f(U) ⊂ f(β^*g-cl(U)) ⊂ f($f^{-1}(B)$). Since f is strongly β^*g-open almost β^*g-irresolute surjection, we obtain A ⊂ f(U) ⊂ β^*g-cl(f(U)) ⊂ B. Then again by **Theorem 3.8,** the space Y is β^*g-normal.

4.7 Theorem. If f : X → Y is an strongly β^*g-closed continuous function from a β^*g-normal space X onto a space Y , then Y is β^*g-normal.

Proof: Let M_1 and M_2 be disjoint closed sets. Then $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are closed sets. Since X is β^*g-normal, then there exist disjoint β^*g-open sets U and V such that $f^{-1}(M_1) \subset U$ and $f^{-1}(M_2) \subset V$. By **Theorem 4.3**, there exist β^*g-open sets A and B such that $M_1 \subset A$, $M_2 \subset B$, $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Also, A and B are disjoint. Thus, Y is β^*g-normal.

5. Some generalized β^*g-closed functions

5.1 Definition. A function $f : X \to Y$ is said to be

(1) **β^*g-closed** if f(A) is β^*g-closed in Y for each closed set A of X,

(2) **β^*gg-closed** if f(A) is β^*gg-closed in Y for each closed set A of X,

(3) **gβ^*g-closed** if f(A) is gβ^*g-closed in Y for each closed set A of X.

5.2 Definition. A function $f : X \to Y$ is said to be

(1) **quasi β^*g-closed** if f(A) is closed in Y for each $A \in \beta^*gC(X)$,

(2) **β^*g-β^*gg-closed** if f(A) is β^*gg-closed in Y for each $A \in \beta^*gC(X)$,

(3) **β^*g-gβ^*g-closed** if f(A) is gβ^*g-closed in Y for each $A \in \beta^*gC(X)$,

(4) **almost gβ^*g-closed** if f(A) is gβ^*g-closed in Y for each $A \in RC(X)$.

5.3 Definition. A function $f : X \to Y$ is said to be **β^*g-gβ^*g-continuous** if $f^{-1}(K)$ is gβ^*g-closed in X for every $K \in \beta^*gC(Y)$.

5.4 Definition. A function $f : X \to Y$ is said to be **β^*g-irresolute** if $f^{-1}(V) \in \beta^*gO(X)$ for every $V \in \beta^*gO(Y)$.

5.5 Theorem. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then

(1) the composition $gof : X \to Z$ is β^*g-gβ^*g-closed if f is β^*g-gβ^*g-closed and g is continuous β^*g-gβ^*g-closed.

(2) the composition $gof : X \to Z$ is β^*g-gβ^*g-closed if f is strongly β^*g-closed and g is β^*g-gβ^*g-closed.

(3) the composition $gof : X \to Z$ is β^*g-gβ^*g-closed if f is quasi β^*g-closed and g is g β^*g-closed.

5.6 Theorem. Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition $gof : X \to Z$ be β^*g-gβ^*g-closed. If f is a β^*g-irresolute surjection, then g is β^*g-gβ^*g-closed.

Proof: Let $K \in \beta^*gC(Y)$. Since f is β^*g-irresolute and surjective, $f^{-1}(K) \in \beta^*gC(X)$ and $(gof)(f^{-1}(K)) = g(K)$. Hence, g(K) is gβ^*g-closed in Z and hence g is β^*g-gβ^*g-closed.

5.7 Lemma. A function $f : X \to Y$ is β^*g-gβ^*g-closed if and only if for each subset B of Y and each $U \in \beta^*gO(X)$ containing $f^{-1}(B)$, there exists a g β^*g-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

Proof: ($\Rightarrow$) : Suppose that f is β^*g-gβ^*g-closed. Let B be a subset of Y and $U \in \beta^*gO(X)$ containing $f^{-1}(B)$. Put $V = Y - f(X - U)$, then V is a g β^*g-open set of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

($\Leftarrow$) : Let K be any β^*g-closed set of X. Then $f^{-1}(Y - f(K)) \subset X - K$ and $X - K \in \beta^*gO(X)$. There exists a gβ^*g-open set V of Y such that $Y - f(K) \subset V$ and $f^{-1}(V) \subset X - K$. Therefore, we have $f(K) \supset Y - V$ and $K \subset f^{-1}(Y - V)$. Hence, we obtain $f(K) = Y - V$ and f(K) is gβ^*g-closed in Y . This shows that f is β^*g-gβ^*g-closed.

5.8 Theorem. If $f : X \to Y$ is continuous β^*g-gβ^*g-closed, then f(H) is gβ^*g-closed in Y for each gβ^*g-closed set H of X.

Proof: Let H be any gβ^*g-closed set of X and V an open set of Y containing f(H). Since $f^{-1}(V)$ is an open set of X containing H, $\beta^*g\text{-}cl(H) \subset f^{-1}(V)$ and hence $f(\beta^*g\text{-}cl(H)) \subset V$. Since f is β^*g-gβ^*g-closed and $\beta^*g\text{-}cl(H) \in \beta^*gC(X)$, we have $\beta^*g\text{-}cl(f(H)) \subset \beta^*g\text{-}cl(f(\beta^*g\text{-}cl(H))) \subset V$. Therefore, f(H) is g$\beta^*$g-closed in Y .

5.9 Remark. Every β^*g-irresolute function is β^*g-gβ^*g-continuous but not conversely.

5.10 Theorem. A function $f : X \rightarrow Y$ is β^*g-gβ^*g-continuous if and only if $f^{-1}(V)$ is gβ^*g-open in X for every $V \in \beta^*gO(Y)$.

5.11 Theorem. If $f : X \rightarrow Y$ is closed β^*g-gβ^*g-continuous, then $f^{-1}(K)$ is gβ^*g-closed in X for each gβ^*g-closed set K of Y .

Proof: Let K be a gβ^*g-closed set of Y and U an open set of X containing $f^{-1}(K)$. Put $V = Y - f(X - U)$, then V is open in Y , $K \subset V$, and $f^{-1}(V) \subset U$. Therefore, we have $\beta^*g\text{-}cl(K) \subset V$ and hence $f^{-1}(K) \subset f^{-1}(\beta^*g\text{-}cl(K)) \subset f^{-1}(V) \subset U$. Since f is β^*g-gβ^*g-continuous, $f^{-1}(\beta^*g\text{-}cl(K))$ is gβ^*g-closed in X and hence $\beta^*g\text{-}cl(f^{-1}(K)) \subset \beta^*g\text{-}cl(f^{-1}(\beta^*g\text{-}cl(K))) \subset U$. This shows that $f^{-1}(K)$ is gβ^*g-closed in X.

5.12 Corollary. If $f : X \rightarrow Y$ is closed β^*g-irresolute, then $f^{-1}(K)$ is gβ^*g-closed in X for each gβ^*g-closed set K of Y .

5.13 Theorem. If $f : X \rightarrow Y$ is an open β^*g-gβ^*g-continuous bijection, then $f^{-1}(K)$ is gβ^*g-closed in X for every gβ^*g-closed set K of Y .

Proof: Let K be a gβ^*g-closed set of Y and U an open set of X containing $f^{-1}(K)$. Since f is an open surjective, $K = f(f^{-1}(K)) \subset f(U)$ and f(U) is open. Therefore, $\beta^*g\text{-}cl(K) \subset f(U)$. Since f is injective, $f^{-1}(K) \subset f^{-1}(\beta^*g\text{-}cl(K)) \subset f^{-1}(f(U)) = U$. Since f is β^*g-gβ^*g-continuous, $f^{-1}(\beta^*g\text{-}cl(K))$ is gβ^*g-closed in X and hence $\beta^*g\text{-}cl(f^{-1}(K)) \subset \beta^*g\text{-}cl(f^{-1}(\beta^*g\text{-}cl(K))) \subset U$. This shows that $f^{-1}(K)$ is gβ^*g-closed in X.

5.14 Theorem. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions and let the composition $gof : X \rightarrow Z$ be β^*g-gβ^*g-closed. If g is an open β^*g-gβ^*g-continuous bijection, then f is β^*g-gβ^*g-closed.

Proof: Let $H \in \beta^*gC(X)$. Then (gof)(H) is gβ^*g -closed in Z and $g^{-1}((gof)(H)) = f(H)$. By **Theorem 5.13**, f(H) is g β^*g-closed in Y and hence f is β^*g-gβ^*g-closed.

5.15 Theorem. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions and let the composition $gof : X \rightarrow Z$ be β^*g-gβ^*g-closed. If g is a closed β^*g-gβ^*g-continuous injection, then f is β^*g-gβ^*g-closed.

Proof: Let $H \in \beta^*gC(X)$. Then (gof)(H) is gβ^*g-closed in Z and $g^{-1}((gof)(H)) = f(H)$. By **Theorem 5.11**, f(H) is gβ^*g-closed in Y and hence f is β^*g-gβ^*g-closed.

6. Preservation theorems and other characterizations of β^*g-normal spaces

6.1 Theorem. For a topological space X, the following are equivalent :

(a) X is β^*g-normal,

(b) for any pair of disjoint closed sets A and B of X, there exist disjoint gβ^*g-open sets U and V of X such that $A \subset U$ and $B \subset V$,

(c) for each closed set A and each open set B containing A, there exists a gβ^*g-open set U such that $cl(A) \subset U \subset \beta^*g\text{-}cl(U) \subset B$,

(d) for each closed A and each g-open set B containing A, there exists a β^*g-open set U such that $A \subset U \subset \beta^*g\text{-}cl(U) \subset int(B)$,

(e) for each closed A and each g-open set B containing A, there exists a gβ^*g-open set G such that $A \subset G \subset \beta^*g\text{-}cl(G) \subset int(B)$,

(f) for each g-closed set A and each open set B containing A, there exists a β^*g-open set U such that $cl(A) \subset U \subset \beta^*g\text{-}cl(U) \subset B$,

(g) for each g-closed set A and each open set B containing A, there exists a gβ^*g-open set G such that $cl(A) \subset G \subset \beta^*g\text{-}cl(G) \subset B$.

Proof: (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) : Since every β^*g-open set is gβ^*g-open, it is obvious.

(d) $\Rightarrow$ (e) $\Rightarrow$ (c) and (f) $\Rightarrow$ (g) $\Rightarrow$ (c) : Since every closed (resp. open) set is g-closed (resp. g-open), it is obvious.

(c) $\Rightarrow$ (e) : Let A be a closed subset of X and B be an g-open set such that $A \subset B$. Since B is g-open and A is closed, $A \subset int(A)$. Then, there exists a $g\beta^*g$-open set U such that $A \subset U \subset \beta^*g\text{-}cl(U) \subset int(B)$.

(e) $\Rightarrow$ (d) : Let A be any closed subset of X and B be a g-open set containing A. Then there exists a $g\beta^*g$-open set G such that $A \subset G \subset \beta^*g\text{-}cl(G) \subset int(B)$. Since G is $g\beta^*g$-open, $A \subset \beta^*g\text{-}int(G)$. Put $U = \beta^*g\text{-}int(G)$, then U is β^*g-open and $A \subset U \subset \beta^*g\text{-}cl(U) \subset int(B)$.

c) $\Rightarrow$ (g) : Let A be any g-closed subset of X and B be an open set such that $A \subset B$. Then $cl(A) \subset B$. Therefore, there exists a $g\beta^*g$-open set U such that $cl(A) \subset U \subset \beta^*g\text{-}cl(U) \subset B$.

(g) $\Rightarrow$ (f) : Let A be any g-closed subset of X and B be an open set containing A. Then there exists a $g\beta^*g$-open set G such that $cl(A) \subset G \subset \beta^*g\text{-}cl(G) \subset B$. Since G is $g\beta^*g$-open and $cl(A) \subset G$, we have $cl(A) \subset \beta^*g\text{-}int(G)$, put $U = \beta^*g\text{-}int(G)$, then U is β^*g-open and $cl(A) \subset U \subset \beta^*g\text{-}cl(U) \subset B$.

6.2 Theorem. If $f : X \to Y$ is a continuous quasi β^*g-closed surjection and X is β^*g-normal, then Y is normal.

Proof: Let M_1 and M_2 be any disjoint closed sets of Y . Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X. Since X is β^*g-normal, there exist disjoint $U_1, U_2 \in \beta^*gO(X)$ such that $f^{-1}(M_i) \subset U_i$ for i = 1, 2. Put $V_i = Y - f(X - U_i)$, then V_i is open in Y, $M_i \subset V_i$ and $f^{-1}(V_i) \subset U_i$ for i = 1, 2. Since $U_1 \cap U_2 = \phi$ and f is surjective; we have $V_1 \cap V_2 = \phi$. This shows that Y is normal.

6.3 Lemma. A subset A of a space X is $g\beta^*g$-open if and only if $F \subset \beta^*g\text{-}int(A)$ whenever F is closed and $F \subset A$.

6.4 Theorem. Let $f : X \to Y$ be a closed β^*g-$g\beta^*g$-continuous injection. If Y is β^*g-normal, then X is β^*g-normal.

Proof: Let N_1 and N_2 be disjoint closed sets of X, Since f is a closed injection, $f(N_1)$ and $f(N_2)$ are disjoint closed sets of Y . By the β^*g-normality of Y , there exist disjoint $V_1, V_2 \in \beta^*gO(Y)$ such that $f(N_i) \subset V_i$ for i = 1, 2. Since f is β^*g-$g\beta^*g$-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint $g\beta^*g$-open sets of X and $N_i \subset f^{-1}(V_i)$ for i = 1, 2. Now, put $U_i = \beta^*g\text{-}int(f^{-1}(V_i))$ for i = 1, 2. Then, $U_i \in \beta^*gO(X)$, $N_i \subset U_i$ and $U_1 \cap U_2 = \phi$. This shows that X is β^*g-normal.

6.5 Corollary. If $f : X \to Y$ is a closed β^*g-irresolute injection and Y is β^*g-normal, then X is β^*g-normal.

Proof: This is an immediate consequence since every β^*g-irresolute function is β^*g-$g\beta^*g$-continuous.

6.6 Lemma. A function $f : X \to Y$ is almost $g\beta^*g$-closed if and only if for each subset B of Y and each $U \in RO(X)$ containing $f^{-1}(B)$, there exists a $g\beta^*g$-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

6.7 Lemma. If $f : X \to Y$ is almost $g\beta^*g$-closed, then for each closed set M of Y and each $U \in RO(X)$ containing $f^{-1}(M)$, there exists $V \in \beta^*gO(Y)$ such that $M \subset V$ and $f^{-1}(V) \subset U$.

6.8 Theorem. Let $f : X \to Y$ be a continuous almost $g\beta^*g$-closed surjection. If X is normal, then Y is β^*g-normal.

Proof: Let M_1 and M_2 be any disjoint, closed sets of Y. Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X. By the normality of X, there exist disjoint open sets U_1 and U_2 such that $f^{-1}(M_i) \subset U_i$, where i = 1, 2. Now, put $G_i = int(cl(U_i))$ for i = 1, 2, then $G_i \in RO(X)$, $f^{-1}(M_i) \subset U_i \subset G_i$ and $G_1 \cap G_2 = \phi$. By **Lemma 6.7**, there exists $V_i \in \beta^*gO(Y)$ such that $M_i \subset V_i$ and $f^{-1}(V_i) \subset G_i$, where i = 1, 2. Since $G_1 \cap G_2 = \phi$ and f is surjective, we have $V_1 \cap V_2 = \phi$. This shows that Y is β^*g-normal.

6.9 Corollary. If $f : X \to Y$ is a continuous β^*g-closed surjection and X is normal, then Y is β^*g-normal.

References

1. M. E. Abd EI-Monsef, S. N. EI Deeb and R. A. Mohamoud, β -open sets and β-continuous mappings, Bull. Fac. Assiut Univ. Sci., **12**(1983), 77-90.

2. M. E. Abd EI-Monsef, R. A. Mahmoud, and E. R. Lashin, , β-closure and β-interior, J. Fac. Edu. Ain Shams Univ., **10**(1986), 235.

3. D. Andrijevic, on b-open sets, Mat. Vesnik, 48 (1996), 59-64.

4. S. P. Arya and R. Gupta, On strongly continuous functions, Kyungpook Math. J., 14 (1974), 131-141.

5. S. P. Arya and T. Nour, Characterizations of s-normal spaces, Indian J. Pure Appl. Math., 21 (1990), 717-719.

6. P. Bhattacharyya and B. K. Lahiri, Semi generalized closed sets in topology, Indian J. Math., 29 (1987), 375-382.

7. K. Balachandran, P. Sundram, H. Maki and A. Rani, On generalized preclosed sets, preprint.

8. S. S. Benchalli and P. G. Patil, Some new continuous maps in topological spaces, Journal of Advanced Studies in Topology, 2/1-2, (2009), 53-63.

9. D. Carnahan, Some properties related to compactness in topological spaces, PhD thesis, University of Arkansas, 1973.

10. S. G. Crossley and S. K. Hildebrand, Semiclosure, Texas J. Sci., 22 (1971), 99-112. On β*g-normal spaces 271

11. J. Dontchev, On generalized semi-preopen sets, Mem. Fac. Sci. Kochi Univ. (Math.), **16**(1995), 35.

12. J. Dontchev and M. Przemski, On the various decompositions of continuous and some weakly continuous functions, Acta Math. Hungar., 71 (1-2) (1996), 109-120.

13. A. A. El-Atik, A study of some types of mappings on topological spaces, M. Sci. thesis, Tanta Univ., Egypt, 1997.

14. E. Ekici, On γ-normal spaces, Bull. Math. Soc. Sci. Math. Roumanie Tome, 50(98), **3**(2007), 259-272.

15. D. S. Jankovic´, A note on mappings of extremally disconnected spaces, Acta Math. Hungar., 46 (1-2) (1985), 83-92.

16. N. Levine, Semiopen sets and semicontinuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.

17. N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo, 19 (2) (1970), 89-96.

18. S. N. Maheshwari and R. Prasad, On s-normal spaces, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.), 22 (68) (1978), 27-29.

19. R. A. Mahmoud and El-Monsef, M. E., β-irresolute and β-topological invariant, Proc. Pakistan Acad. Sci., 27(1990), 285.

20. H. Maki, J. Umehara and T. Noiri, Every topological space is pre-$T_{1/2}$, Mem. Fac. Sci. Kochi Univ. Ser. A Math., 17 (1996), 33-42.

21. H. Maki, R. Devi and K. Balachandram, Associated topologies of generalized α-open and α-generalized closed sets, Mem. Fac. Sci. Kochi Univ. Math.**1**(1994), 51-63.

22. H. Maki, T. Noiri and M. Hussain, On s*g-closed sets and s*-normal spaces, JNSMAC, Vol. 48, No. 1 & 2, (2008), 31-41.

23. A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, 53 (1982), 47-53.

24. O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15 (1965), 961-970.

25. T. Noiri, Semi-normal spaces and some functions, Acta Math. Hungar., 65 (3) (1994), 305-311.

26. T. Noiri, H. Maki and J. Umehera, Generalized preclosed functions, Mem. Fac. Sci. Kochi Univ. (Math.), 19(1998), 13-20.

27. T. Noiri, Almost αg-closed functions and separation axioms, Acta Math. Hungar., 82 (3) (1999), 193-205.

28. T. M. J. Nour, Contributions to the theory of bitopological spaces, PhD thesis, Delhi University, India, 1989.

29. Paul and Bhattacharyya, On p-normal spaces, Soochow J. Math., 21(3), (1995), 273-289.

30. J. H. Park, Y. B. Park and B. Y. Lee, On gp-closed sets and pre gp-continuous functions, Indian J. Pure Appl. Math., 33(1), (2002), 3-12.

31. M. C. Sharma and H. Kumar, On β^{*}g-closed sets and β^{*}-normal spaces, Acta Ciencia Indica, Vol. XLI, M. No. 1, (2015), 67-72.

32. M. H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., **41**(1937), 375-381.