THE INFLUENCE OF RELATIVITY ON KEPLAR PROBLEM

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ABSTRACT

Taking help of an iteration process to create from Dirac first order equation for the electron in a coulomb field. We bring out the cause why Schrödinger's relativistic equation failed to come the solution for Kepler problem.

Introduction

The completion of sixty years of the Dirac equation is a good occasion for stock taking of the Vissicitudes in physical theory Since the birth of Quantum Mechanics with regard to covariance space and time were treated by Dirac on the same footing his equation contains first order derivatives with respect to each space co-ordinate and time. Schrödinger gave a second order equation (i.e) an equation containing second order derivatives of the wave function in space and time. In this paper we shall try to relate Schrödinger second order relativistic equation with the Dirac equation with special reference to the Keplar problem (l.e) the problem of an electron in a coulomb field.

Schrödinger's Relativistic Equation:-

For a free electron with momentum p and energy E we can write

$$(C^2\ P^2 + m^2 + c^4)\ \psi = E^2\ \psi \ldots\ldots\ldots\ldots(1)$$

Where m is the rest mass of the electron C the velocity of light and ψ the plane wave for the electron. From this we get the second order wave equation.

$$-\hbar^2 c^2\ \nabla^2\ \psi + m^2 c^4\ \psi = -h2\ \frac{\partial^2\psi}{\partial t^2} \ldots\ldots\ldots\ldots(2)$$

Its solution may be written in the form

$$\psi = \exp\{i(k.r - wt)\},\ \text{with energy}$$

$$E = \hbar w = \pm(\hbar^2 c^2 k^2 + m^2 c^4)^{1/2}$$

The signs + and – represent electrons and positrons respectively which is a non-trivial physical input from Dirac's theory.

we can now go over to the case of the electron in a coulomb field. we introduce the four potential $A\big((r,t),\ \emptyset\ (r,t)\big)$ where $\emptyset$ and 1/c A have the same covariance behavior as E and P. So, we write

$$(E - e\emptyset)^2 = (c\ p - eA)^2 + m^2 c^4 \ldots\ldots\ldots\ldots(3)$$

Now let A(r,t)=o and $\emptyset$ be the both time independent and Spherically symmetric writing

ψ (r,t)= u(r) exp $\left(-iE\, {}^{t}/_{\hbar}\right)$ and

u(r)= R(r)y_l^m $(\theta, \emptyset)$ we get the radial Schrödinger equation for the relativistic electron

$\left[-\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr}) + \frac{l(l+1)}{r^2}\right]R(r) = \frac{(E-e\phi)^2-m^2c^2}{\hbar^2c^2}$R (where l = 0,1,2,......)

We can now specialize to the coulomb field

e$\emptyset = \frac{-ze^2}{r}$, z being the nuclear charge, we get finally

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{2k\eta}{r} + k^2 - \frac{l(l+1) -\propto^2 z^2}{r^2}\right]R = O$$

where $\propto = {e^2}/_{\hbar c}$ the fine structure constant and $\eta = \frac{\propto ZE}{\hbar ck}$, the relativistic analogue of sommerfeld's parameter ${ze^2}/_{\hbar v}$ note the similarity of this equation with the non-relativistic Schrödinger coulomb equation.

We have a non-integral orbital angular momentum

$$l(l+1) \rightarrow l(l+1) -\propto^2 z^2$$

where $\propto$ is the fine structure constant (i.e) coupling of the electron to the coulomb field.

The energy levels are given by

$$\text{E}= mc^2\left(1+\frac{\propto^2 z^2}{{\eta_b}^2}\right), \eta_B = \frac{\propto ZE}{\hbar ck_B}$$

η_B is the bound state similarity of sommerfeld's parameter.

$\eta_\beta = n' + \gamma + 1$ where n' is the positive integer or 0, and γ is a non negative solution of the equation.

$$\gamma(\gamma+1) = l(l+1) -\propto^2 z^2$$

in other words,

$\gamma =-{}^1/_2 \pm {}^1/_2\, [(2l+1)^2 - 4 \propto^2 z^2]^{1/2}$

For l =o both the solution are negative the choice being based on boundary condition at r = o

Two obvious limiting cases of the relativistic Schrödinger coulomb equation

i) Plane wave limit : z = 0, and $\eta = 0$ and $\boldsymbol{l}$ is an integer
ii) Non relativistic coulomb case : E$\gg$ $\hbar ck_\beta$ (i.e) E$\rightarrow$ mc^2, so that $\propto^2 z^2 \ll l(l+1)$

the term remain in equation (5), but $\boldsymbol{l}$ again becomes an integer. To use Sommerfeld'd language, the semi-classical orbit of the electron gets closed and there is no precession. Thus, the Schrödinger coulomb relativistic equation indicates a non-integer orbital angular momentum due to coupling of the electron to the coulomb field.

It yields the non relativistic Schrödinger coulomb equation in the appropriate limit, but it does not explicitly contain electron spin.

Dirac introduced electron spin as well as negative energy states in a natural way. We can see this by writing out the differential equation with generalized co-efficient, (i.e) without prejudice to the nature of the co-efficient.

$$\mathcal{D}_D\psi \cong [c \propto .p + \beta_m c^2]\psi = i\hbar \frac{\partial\psi}{\partial t}$$

$\mathcal{D}_{\mathcal{D}}$ being the Dirac plane wave operator. On iterating with $\mathcal{D}_{\mathcal{D}}$ we should get the identity

$$c^2p^2 + m^2c^4 = E^2$$

this is the commutation relations for the Dirac matrices :

$$\propto_i\propto_j + \propto_j\propto_i = 2\delta_{ij}$$

$$\propto_i \beta + \beta \propto_i = O$$

A minimal realization of the anti-commutators may be obtained by 4X4 matrices of the type

$$\propto_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} I_2 & O \\ o & -I_2 \end{pmatrix}$$

Where σ are the well known pauli spin matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

For convenience, we use another set of Dirac matrices $\rho_{i(i=1,2,3)}$ and I, the 4x4 unit matrix,

Where $\propto_i = p_1\sigma_i$,

$\beta = \rho_3 I_2$ Here I_2 is

the 2 x 2 unit matrix.

The ρ matrices have the same form in energy space as the Pauli matrices have in spin space. The Dirac plane wave equation takes the form

$O_+\psi \equiv \left[\rho_2\sigma.\nabla - \rho_3 \ {}^{E}/_{\hbar c} + {}^{mc}/_{\hbar}\right] \psi = 0$………….(A)

Assuming, $O_- \equiv O_+ - 2\ {}^{mc}/_{\hbar}$

Then, since $(\rho_2\sigma.\nabla)^2 = \nabla^2$,

$$O_-O_+\emptyset = \left[\frac{1}{r^2}\ \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{l^2}{r^2} + k^2\right]\emptyset = 0$$

………(B)

Equation (B) is the second order equation corresponding to the first order eqn(A) note that spin is not explicit in this equation and that positive and negative energy States have been clubbed together by the process of iteration.

Also, every soln of eqn (A) satisfies eqn (B), But not vice versa,

Hence we use a different symbol ∅ to denote solution of equation (B). We can now bring out the spin dependence of these solutions by writing

$$l^2 = (\sigma.L+1)^2 - (\sigma.L+1) \text{..........(C)}$$

It is once seen that the constants of the motion for the second order equation are ρ_3 and the Dirac operator K=$\rho_3(\sigma.L+1)$, using the pauli spinor.

$x_{1/2}^{1/2} = \binom{1}{0}$ and $x_{1/2}^{-1/2} = \binom{0}{1}$

We can introduced Spinor solutions,

$$x_k^m = \sum_T c \begin{matrix} u-\mathcal{TTM} \\ i\,{}^1\!/_2\, t \end{matrix} Y_l^{\mathcal{M}-\mathcal{T}}(\theta,\emptyset) X_{1/2}^{\mathcal{T}} \text{............(D)}$$

which satisfy the equations

$$(\sigma.L+1)x_k^{\mathcal{M}} = -kx_k^{\mathcal{M}} \text{...........(E)}$$

$$\sigma.r \;\; x_{-k}^{\mathcal{M}} = x_k^u$$

From equation © we see that the orbital angular momentum l is given by $l(l+1) = k(k+1)$, so that $l(k) = IKI + \frac{1}{2}[sgn(k) - 1]$..............(F)

The plane wave soln with sharp K and p_3 *are*

$$\emptyset_+ \equiv \emptyset_{\rho 3=+1} = \left(j_{l(-k)}(k\;\;r)X_{-k}^u\right) \text{............(G)}$$

$$\emptyset_- \equiv \emptyset_{\rho 3=-1} = \left(j_{l(k)}(k\;\;r)X_k^u\right) \text{..............(H)}$$

Now, Since $[O_+ O_-] = 0$ obviously

$O_+O_-\;\emptyset = O_-O_+\;\emptyset = 0$ from eqn (B).

In other words, the function O_- ∅statistics the first order equation (A),

Hence we can write

$O_-\;\;\emptyset$ = A ψ, A being a a numrical factor which we can suppras for the moment, For $\rho_3 = -1$, we can write

$$\psi = O_-\;\emptyset_- = \left(\rho\sigma.\nabla - \frac{\rho_3 E}{\hbar c} - mc/\hbar\right)\emptyset_- \text{..........(1)}$$

Since,

$$\sigma.\nabla = \sigma.r\left[\frac{\partial}{\partial r} + \frac{1}{r} - \frac{\sigma.L+1}{r}\right],$$

$$O_{-}=\begin{bmatrix} \frac{(E-MC^2)}{\hbar c} & -i\,\sigma.r\left(\frac{\partial}{\partial r}+\frac{1+K}{r}\right) \\ i\sigma.r\left(\frac{\partial}{\partial r}+\frac{1-K}{r}\right) & -(E+mc^2)/\hbar c \end{bmatrix}$$

Remembering ……the Bssel function

$\left(\frac{\partial}{\partial r}+\frac{1+k}{r}\right)j_{l(k)}(kr) = k\ sgn\ (k)j_{l(-k)(^{kr})}$……(K)

Apart from normalization, this is just the spherical wave solution for the free electron. It has sharp k, j_3 and parity.

The Dirac coulomb wave

The Dirac coulomb wave **is now** obtained by similar way. we start with the first order equation

$O_{+}^{c}\psi \equiv \left[\rho_2\sigma.\nabla - \rho_3\left(\frac{E}{\hbar c}+\frac{\propto Z}{r}\right)+\frac{mc}{\hbar}\right]\psi = 0$………(M)

in which the operator O_{+} differs from the plain wave operator O_{+} only is heving the additional coulomb term$-\rho_3\frac{\propto Z}{r}$. In analogy with equation.

(B), we define

$O_{-}^{c} \equiv O_{+}^{c} - \frac{2mc}{\hbar}$ with which we iterofe equation (M) to get

$$O_{-}^{c}O_{+}^{c}\emptyset = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{\sqcap(\sqcap-1)}{r^2}+\frac{2kn}{r}+k^2\right]\emptyset = 0$$

Where $\sqcap = \rho_3$k+ i$\propto z\rho_1\ \sigma.r$ and

$\sqcap^2 = k^2 - \propto^2 z^2$…………..(O)

Note that $\sqcap$ is the coulomb analogue of $\sigma.L+1$ and that the operators and ρ_3k and $\rho_1\,\sigma.r$ anti commute.

Their major difference lies in the non integral orbital analogue momentum $\sqcap$ related to spin orbit as well as spin coulomb interaction.

The non integral l is different by

$\sqcap \rightarrow \gamma = \pm + I(k^2 - \propto^2 z^2)^{1/2}$ and

$$\sqcap^2 - \sqcap = l(l+1)$$

Explicitly,

L(γ)=1 γ 1+$\frac{1}{2}$[$sgn(\gamma)$-1]…………(P)

In the plane wave limit, γ →k and we get back the Dirac plane wave, comparing Equation (5) and (N) we at once see what was missing in the Schrödinger coulomb equation (5). It was just that instead of the operator $\Pi^2 - \Pi$ it contained $L^2 - \propto^2 z^2 = \Pi^2 - \Pi$ +i $\propto Z\rho\ \sigma r$ in other words, there is an error of the order of the spin couloumb coupling. This messes up the non integral orbital angular momentum. Equation (N) is the correct relativistic extension of the Schrödinger equation for the coulomb potential.

By diagonaliiny Π following the same method as in the Dirac plane wave case. We get the spherical wave solution for the Dirac coulomb electron with sharpk, j,j_3 and parity:

$$\psi = \begin{pmatrix} ik\ \ sgn\ \ (k)F_{l(-\gamma),l}\ \ (kr)\ x_{-k}^{u} \\ -\left[\frac{\left(\frac{KE}{\gamma} + mc^2\right)}{\hbar c}\right] F_{l(\gamma)n}(kr)X_k^u \end{pmatrix}$$

This definite solution corresponds to positive energy States. Π_{n1}(kr) is the well-known radial coulomb wave function.

It will be seen that this method shows as Schrödinger's attempt at a relativistic wave equation missed the point. The key lies in seeing at the first- order equation, which was Dirac's real tour deforce.

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