ON OPERATORS WITH SMALL SELF, COMMUTATORS

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ABSTRACT

Let T be a bounded operator on a Hilbert space H. The self Commutator of T, denoted [T] is T*T-TT*. An operator is of commutator rank n if n the rank of [T] is n. In this Paper operators of commutator rank one are studied. Two particular subclasses are Investigated in detail. T is completely non – normal, or subnormal, if T does not Posses a non – Zero reducting subspace M such that T/M is normal, and T is of Commutator rank n if the rank of [T] is n.

INTRODUCTION

Let H be a Hilbert space. An operater from H to a Hilber space K is understood to Be a bounded linear transformation from H to K.if H=K, the operater is said to be on H. if T is an oparater on H, the self commutator of T , denoted [T], is T*T-TT* is Completetly non – normal, or abnormal, if T does not possess a non – Zero reducting Subspace M such that T/M is norma,and T is of Commutator rank n if the rank of [T] Is n.

Let B (H) denote the set of all operaters on H, and for each non –

Negative integer n,

Let Dn (H) ={T:T€ B (H) and Rank [T] = n},

And E_n(H)= {T:T€ D_n (H) and T*T and TT* commute}.

These last two sets will often be written respectively as D_n and E_n when their

Application to the underlying space H is not to be emphasized . it is immediate

That the classes D_n and E_n consist entirely of normal operators if and only if n

=0. If T is an operator in $D_{1,}$then by multiplying T by a non –Zero real =0.If T is an oprater in D1, then by multiplying T by its adjoint, it may be assumed, without loss of generality, that [T] = p, WHERE p is a one – dimensional self adjoint projection. The purpose of this paper is to study operators T for which [T] has rank one. Some of the theorems stated require the additional condition that T* T and TT* commute.

Theoren(1) Let T is an operator on H. if K is the smallest reducing subspace of T containing the rang of [T], then T/K is the completely non – normal component of T.

Proof:-Let K be as defined in the statement of this theorem by J.W DELALLE[1,TH1] T=T1+T2 onM⊕pM , where T1 is completely non-normal and T2 is normal.

since[T]=[T1]+[since[T]=[T_1]+[T_2],[T_2]=0 and so it is clear that K⊆M,because M is a reducing subspace of T containing the range of[T] then T_1 itself could be furtherreduce into $T_{11}+T_{12}$ on $K+k^1$ where K^1 is the orthogonal complement of K in M. but the definition of K implies that$[T_{12}]=0$,which is contradicts the fact that T_1 is completely non-normal. Therefore K=M.

Theorem(2) Let T£B(H) and suppose that T has the factorization $T=U\sqrt{A}$,where U is unitary and $A=T^*T$.(it is not being assumed that Ker (A)=Ker (U): that is, it is not necessary that $U\sqrt{A}$ be the canonical polar factorization of T.) suppose $T£E_1$ then there is a reducting subspace M of T such that T/M has matrix $\mathcal{M}$ and T/M is normal.

Proof:- Let[T]=P,aProjection of rank one.

$$\text{Then } T^*T-TT^*=P, \therefore \qquad \perp$$

$$\text{So} \quad \sqrt{A}U^*U\sqrt{A}-U\sqrt{A}\sqrt{A}U^*=P,$$

$$\Longrightarrow \quad A-UAU^*=P \qquad (1)$$

U is unitary Since TϵE',T*T and TT* commute, so that P commutes with both T*T and TT*.

Let e be a unit vecter in Range (P). By

Lemma (1), e is an eigenvector for both T*T and TT*. Since both of these operators are non - negative, real scalars a and b such that

$$T^*T\, e = ae$$

$$\text{and } b \quad TT^*e = bc \quad \text{with } a=b+1 \qquad (2)$$

$$\therefore (T^*T - TT^*)\, e = pe \qquad (a-b)e = e \Longrightarrow$$

$$U(T^*T)U^* = TT^* \text{ and } U^*(TT^*)U = T^*T \qquad (3)$$

	T*T	TT	
$(U^*)^n e$	b	b	n>o
e	a	b	
$(U^n)e$	a	b	n>o

(4)

The Proof is by induction on n for n≥0. The case n=0 has been established in (2) Thus suppose n≥0 and suppose the existence of the specified eigen values for this n. Then by (3) and (4)

$$(TT^*)(U^{n+1}e) = (UU^*)(TT^*U)(U^ne)$$
$$=U(U^*TT^*U)U^ne$$
$$=U(T^*T)U^ne$$
$$=U(aU^ne)$$
$$=aU^{n+1}e \qquad (5)$$

$$\text{And } (T^*T)(U^{*n+1}e) = (U^*U)(T^*TU^*)(U^{*n}e)$$

$=U^*(UT^*TU^*)(U^{*n}e)$

$=U^*(TT^*)(U^{*n}e)$

$=U^*(bU^{*n}e)$

$=bU^{*n+1}e$ ________________ (6)

There exit constants c and d such that $pU^{n+1}e=ce$ and $PU^{*n+1}e=de$

Thus,by (5) $T^*T(U^{n+1}e) =(TT^*+p)(U^{n+1}e)$

$=aU^{n+1}e+pu^{n+1}e$

$=aU^{n+1}e+ce$ ______________ (7)

Since p and T*T commute, $p(T^*T)U^{n+1}e =(T^*T)pU^{n+1}e$,

So, $p(aU^{n+1}e+ce) =(T^*T)(ce)$,

$a(pU^{n+1}e)+$

$p(ce) =C(T^*T)e$,

ace+ce cae.

Thus C=0, so that from __________ (7)

$(T^*T)(U^{n+1}e)=aU^{n+1}e$ ____________________ (8)

Observe here that C=0 also implies that $pU^{n+1}e=0$,i.e$pU^{m}e=0$ for all m>0.similarly,by (6),

$TT^*(U^{*n+1}e) =(T^*T-p)(U^{*n+1}e)$

$=bU^{*n+1}e-pU^{*n+1}e$

$=bU^{*n+1}e-de$ ________________ (9)

And Therefore,

$P(T^*T)U^{*n+1}e=(T^*T)PU^{*n+1}e.$

$P(bU^{*n+1}e-de)=(TT^*)(de).$

bde-de =dbe

So that d=0,and(9)=$(TT^*)(U^{*n+1}e)=bU^{*n+1}e$ __________ (10)

The induction is now complete from equation.(5),(6),(8)and(10)

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