A STUDY ON APPLICATION OF RUNGE-KUTTA METHOD WITH RESPECT TO TRAFFIC FLOW PROBLEM

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ABSTRACT

The Runge –Kutta method is a technique of numerically iterating Ordinary Differential Equations (ODE). Nowadays the need for solving real-world problems has been increased. The problem in question here is a traffic flow problem involving different parameters. Solving vehicular traffic flows based on Partial Differential Equations (PDE) are more complicated and time-consuming that cannot be done in real-time, so ODE mathematical models are constructed from obtained PDE. The accuracy of the Runge-Kutta method is more reliable compared to other iteration methods. These accurate solutions can be widely used to solve the increasing need for traffic flow optimization. In this paper, the construction of an ODE mathematical model is done for a real-world problem, in particular, traffic flow is discussed and a numerical solution is given as an example. We have also compared the solution obtained from the RK method with Euler's method, thus showing, the solution obtained from an IV ORDER Runge-Kutta method (for first order ODE) is more accurate.

Keywords: *Runge-Kutta method, Ordinary differential equation, Traffic flow, Partial Differential Equations, Mathematical models.*

INTRODUCTION:

Traffic flow is the study of the movement of individual vehicles between two points for a particular period and the interaction they make with each other, with the aim of understanding and providing an optimal solution. Here we consider a real-time traffic flow problem, which is converted into a mathematical model to obtain an optimal solution and it is used to verify the traffic density between two traffic signals in the city.

The traffic flows are mainly based on the Partial Differential Equations, but it is very complicated to solve and deduce the solution. So to make this model more convenient we use numerical methods to obtain the solution. Numerical differentiation is used for evaluation because it addresses two issues, accuracy and time. Here we consider the Runge –Kutta method as it reduces the time spent and gives relatively more accurate and precise answers. To use this method we convert PDE into ODE, as Runge–Kutta method is applicable only for Ordinary Differential Equations. The IV ORDER Runge–Kutta method (for first order ODE) is more accurate than first, second and third orders Runge–Kutta method.

The real world problem is converted to a mathematical model and the solution to be obtained is discussed below. The outcome of the model gives a numerical solution to vehicular density and velocity of traffic flow between two consecutive signals in real time situation.

CONSTRUCTION OF A TRAFFIC FLOW PROBLEM:

VARIABLES (or) PARAMETERS:

In this paper, we consider, only vehicular density (ρ) and velocity (v) to be the relevant variables though there could be many other factors affecting the system such as accidents, on/off ramps, length of roadway, the spacing between cars, etc. Thus ignoring the other factors, let F(ρ,v) represent the Traffic Flow, in which we define ρ(x,t) as the vehicular density, in particular for car-density and v(x,t) as the velocity at a point x and time t. The mathematical model for the above data will be a Partial Differential Equation (PDE). Hence the obtained PDE will be converted into a pair of Ordinary Differential Equations (ODE) and solved further.

ODE MATHEMATICAL MODEL:

Let the function, F(ρ,v)=0,

Here, ρ is a function of x and t. The PDE is given as follows:

$$\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \qquad \text{……………………. (1)}$$

Let us assume that **ρ(x,t) = U(*x*)W(*t*)**, i.e., ρ can be written as the product of two functions, one depends only on *x*, the other depends only on t. Thus we get,

$$W\frac{dU}{dx} + U\frac{dW}{dt} = 0 \qquad \text{………………………. (2)}$$

The above equation is reconstructed to collect the variable x in one side, and the variable t in the other side.

$$\frac{1}{U}\frac{dU}{dx} = -\frac{1}{W}\frac{dW}{dt} \qquad \text{……………………. (3)}$$

Let us introduce a common constant *"a", such that,*

$$\frac{1}{U}\frac{dU}{dx} = -\frac{1}{W}\frac{dW}{dt} = a \qquad \text{……………………. (4)}$$

The given PDE is now separated to form two ODE's,

$$\frac{1}{U}\frac{dU}{dx} = a \qquad \text{……………………. (5)}$$

$$\frac{1}{W}\frac{dW}{dt} = -a \qquad \text{……………………. (6)}$$

The general solutions for equations (5) and (6) are $U(x) = k_1 \exp(ax)$ *and* $W(t) = k_2 \exp(-at)$, where k_1 and k_2 are arbitrary constants.
Also, equations (5) and (6) can be again reconstructed to form a PDE with general solution.

Similarly, consider v as a function of x and t. The above process is followed to find the general solutions for, $\frac{\partial v}{\partial x}+\frac{\partial v}{\partial t}=B\frac{dA}{dx}+A\frac{dB}{dt}=0$ as, *A(x) = l_1 exp(rx) and B(t) = l_2 exp(-rt)*, where r is a common constant and l_1, l_2 are arbitrary constants.

NUMERICAL SOLUTION:

IV ORDER Runge-Kutta Method:

For a given Ordinary Differential Equation, the IV ORDER R.K method is given by,

$$\mathbf{k_1 = hf\,(x,y)}$$

$$\mathbf{k_2 = hf\,(x+\frac{h}{2}\,,\ y+\frac{k_1}{2})}$$

$$\mathbf{k_3 = hf\,(x+\frac{h}{2}\,,\ y+\frac{k_2}{2})}$$

$$\mathbf{k_4 = hf\,(x+h\,,\ y+k_3)}$$

and $\mathbf{\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4)}$ and $\mathbf{y(x+h) = y(x) + \Delta}$ where, $\mathbf{h = \Delta x}$

ILLUSTRATION:

Consider an ODE of first order for one parameter, say v(t,x).

Let the differential equation be, $\mathbf{x' = t^2+x}$ with initial condition, x(0)=1. Assume h=0.1. Now let us find the value of x(1.0).

From the given, x(0) = 1 ➔ $t_0=0$; $x_0=1$ and h=0.1

By IV ORDER Runge-Kutta method,

$K_1 = hv\,(t_0\,,\,x_0) = 0.1$

$K_2 = hv\,(t_0+\frac{h}{2}\,,\ x_0+\frac{K_1}{2}) = 0.1053$

$K_3 = hv\,(t_0+\frac{h}{2}\,,\ x_0+\frac{K_2}{2}) = 0.1055$

$K_4 = hv\,(t_0+h\,,\ x_0+K_3) = 0.1116$

$\Delta x = \frac{1}{6}(K_1+2K_2+2K_3+K_4) = 0.1055$ and

$$\mathbf{x_1 = x(0.1) = x_0 + \Delta x = 1.1055}$$

Now, we can find x(0.2) with initial values $x_1 = 1.1055$ and $t_1 = 0.1$

$K_1 = hv\,(t_1\,,\,x_1) = 0.1116$

$K_2 = hv\,(t_1+\frac{h}{2}\,,\ x_1+\frac{K_1}{2}) = 0.1184$

$K_3 = hv\,(t_1+\frac{h}{2}\,,\ x_1+\frac{K_2}{2}) = 0.1187$

$K_4 = hv\,(t_1+h\,,\ x_1+K_3) = 0.1264$

$\Delta x = \frac{1}{6}(K_1+2K_2+2K_3+K_4) = 0.1187$ and

$\mathbf{x_2 = x(0.2) = x_1 + \Delta x = 1.2242}$

And continuing the process until the time we need to calculate,

For x (1.0) with initial values $x_9 = 2.7689$ and $t_9 = 0.9$

$K_1 = hv\,(t_9\,,\,x_9) \quad = \quad 0.3579$

$K_2 = hv\,(t_9 + \frac{h}{2}\,,\; x_9 + \frac{K_1}{2}) \quad = \quad 0.3850$

$K_3 = hv\,(t_9 + \frac{h}{2}\,,\; x_9 + \frac{K_2}{2}) \quad = \quad 0.3864$

$K_4 = hv\,(t_9 + h\,,\; x_9 + K_3) \quad = \quad 0.4155$

$\Delta x = \frac{1}{6}(K_1+2K_2+2K_3+K_4) \quad = \quad 0.3861$ and

$\mathbf{x_{10} = x(1.0) = x_9 + \Delta x\ 3.1550}$

The value of x_{10} represents, at **t = 1.0** second, the distance covered is **x = 3.1550 meters**. Therefore the **velocity of traffic flow will be, V = 3.155 m/s.**

A similar process can be followed to find the vehicular density, assuming N to be the Number of vehicles with a distance between the signals as 1000 meters (i.e, 1 km) for a particular period of time.

Consider the same problem, to find the solution using Euler's method.

$\mathbf{x(0.1)} = \mathbf{x_1} = x_0 + hv\,(t_0\,,\,x_0) \quad = \quad \mathbf{1.1000}$

$\mathbf{x(0.2)} = \mathbf{x_2} = x_1 + hv\,(t_1\,,\,x_1) \quad = \quad \mathbf{1.2110}$

……

$\mathbf{x(1.0)} = \mathbf{x_{10}} = x_9 + hv\,(t_9\,,\,x_9) \quad = \quad \mathbf{2.9406}$

The solution obtained using Euler's method varies when compared to IV ORDER R.K method. A graphical solution, comparing R.K method and Euler's method is shown below.

GRAPHICAL SOLUTION:

The values obtained from Euler's and RK method are tabulated below:

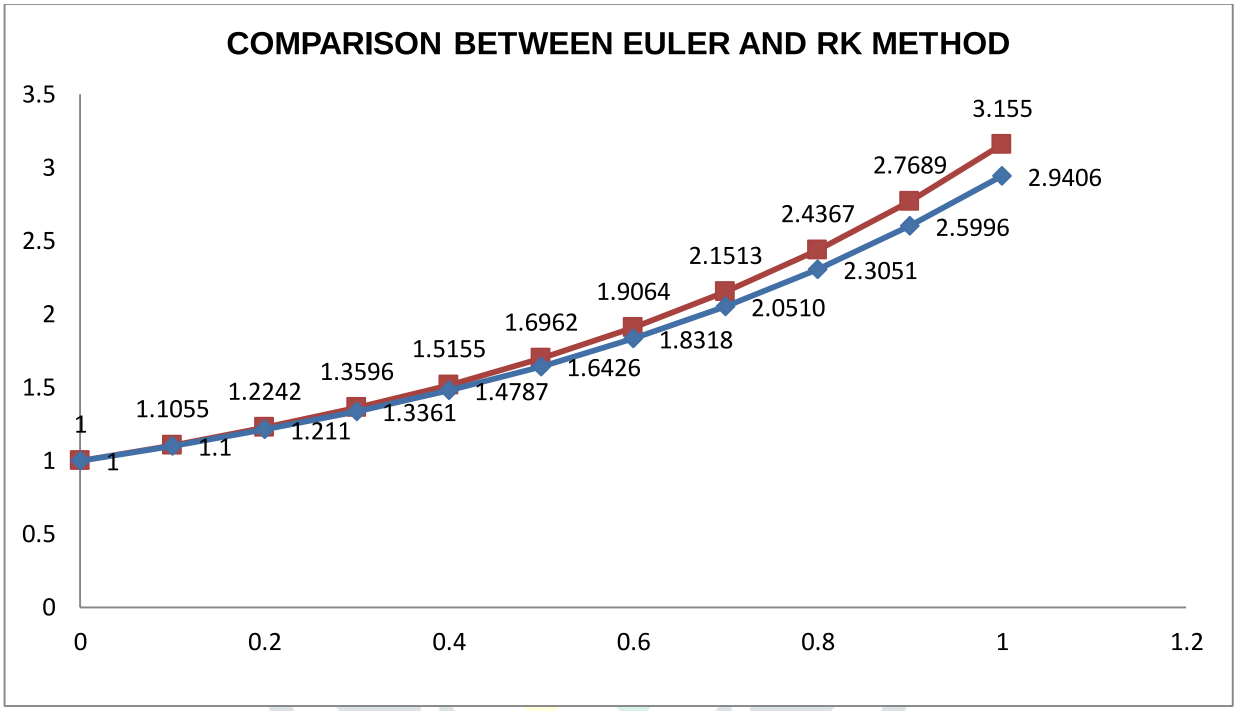
Table 1.1 - COMPARISON OF ACCURACY BETWEEN EULER AND RK METHOD

METHOD / x^n	RK METHOD	EULER'S METHOD
x_0	1	1
x_1	1.1055	1.1
x_2	1.2242	1.211
x_3	1.3596	1.3361
x_4	1.5155	1.4787
x_5	1.6962	1.6426

x_6	1.9064	1.8318
x_7	2.1513	2.0510
x_8	2.4367	2.3051
x_9	2.7689	2.5996
x_{10}	3.155	2.9406

The graph provides comparison of accuracy between RK method and Euler's method.

Table 1.2 – COMPARISON GRAPH BETWEEN EULER AND RK METHOD



From the above table, we observe that the values obtained from Euler's and RK method are not the same. The values distort showing that RK method is more accurate than usual Euler's method.

APPLICATIONS:

The applications of the Runge-Kutta method has a greater effect in areas such as,

Kinetic magnetic induction system

Navier-Stokes system

ADVANTAGES AND LIMITATIONS:

Advantages:

It is mainly used to solve a complex problem, physically or geometrically.

The numerical approach enables the solution of a complex problem with a great number of simple operations.

Time consumption is comparatively low when compared to other analytical methods

The solution obtained is more accurate and precise.

Disadvantages:

- There is a chance for an error of approximation while obtaining a numerical solution.
- Macroscopic problems are complicated to convert into ODE.
- Arriving numerical integrals through computational methods are always expensive.

CONCLUSION:

Traffic flow is the study of movement between two vehicles and the complications in those flow is identified and it is converted into a mathematical model to obtain the mathematical solution. Most of the traffic flow problems that use partial differential equations to deduce the solution but it is complicated to solve the equations and so we use Numerical methods to solve the problem where PDE is converted into ODE and then numerical integration is used as it reduces the time consumption. We use IV ORDER Runge –Kutta method (for first order ODE) which is more accurate and reliable. We have also provided a comparative study between the reliability of the RK method and Euler's method. The procedure looks simple, but more complicated situations arise when boundary conditions are introduced. The above traffic flow analysis gives the traffic density between two traffic signals in the city and it is applied to verify the density and gives the appropriate time to pass the signal and can be used to find the shortest route to reach the destination. This method can also be converted into program code and the solution is obtained within a short period. Thus Runge –Kutta method is the best alternative method to solve the Traffic Flow Analysis for a city to obtain an accurate solution.

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