SOME RESULTS ON DISJUNCTIVE TOTAL DOMINATION NUMBER

[1]G. SANGEETHA,[2]S. VETRISELVI

[1]Assistant Professor, [2]Assistant Professor
[1]Department of Mathematics,
[1]Sri Kaliswari College, Sivakasi, Tamilnadu, India.

Abstract : In this paper, we discuss about the relationship between 2-dominating set, *b*-disjunctive dominating set and disjunctive total dominating set. We prove the realization theorems based on b-disjunctive total domination number and domination number. Also, we determine the value of *b*-disjunctive total domination number for some graphs especially for hypercube.

***IndexTerms* – domination number, *b*-disjunctive domination number, *b*-disjunctive total domination number, hypercube.**

I. INTRODUCTION

A set *S* dominates vertex *v* if *v* is either in *S* or adjacent to (joined by an edge to) some vertex of *S*. For a graph *G*, a set $S \subseteq V(G)$ is a dominating set [5] of *G* if every vertex not in *S* is adjacent to *S*. The domination number, γ (G), is the minimum cardinality of a dominating set. In 2009, Dankelmann et al. [3] concentrated the case where the domination of a vertex reduces as distance [1] increases. Motivated by these ideas, the concept of *b*-disjunctive dominating sets was introduced by Goddard et al.[4] in 2014. To extend this concept with total dominating set, Henning et al. [6] established the concept of disjunctive total domination in 2016 and at the same year, Pandey et al. [7] developed the concept of *b*-disjunctive total domination. First we see about some known parameters.

Definition 1.1. [2] Let G_1 and G_2 be two graphs with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. If $|V_1| = p_1$ and $|V_2| = p_2$, then their *corona* $G_1 \circ G_2$ is obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

Definition 1.2. [2] Let G_1 and G_2 be two graphs with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their *cartesian product* $G_1 \square G_2$ has vertex set $V(G_1) \times V(G_2)$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \square G_2$ whenever $\{u_1 = v_1 \text{ and } u_2 v_2 \in E(G_2)\}$ *or* $\{u_2 = v_2 \text{ and } u_1 v_1 \in E(G_1)\}$.

Definition 1.3. [2] The *hypercube* or *n-cube* Q_n is defined recursively by $Q_1 = K_2$ and $Q_n = K_2 \square Q_{n-1}$.

Definition 1.4. [3] A set $S \subseteq V(G)$ of vertices in a graph *G* is called a *dominating set* if every vertex $v \in V(G)$ is either an element of *S* or adjacent to an element of *S*. A dominating set *S* is a *minimal dominating set* of *G* if no proper subset $S' \subset S$ is a dominating set. The minimum cardinality of a dominating set of *G* is called the *domination number* of *G*. It is denoted by $\gamma(G)$ and the corresponding dominating set is called a γ- *set* of *G*.

Definition 1.5. [4] For a graph *G*, a set $S \subseteq V(G)$ is a *b-dominating set* of *G* if every vertex *v* not in *S* has at least *b* neighbours in *S*. The minimum cardinality of a *b*-dominating set is the *b-domination number* of *G*. It is denoted by $\gamma_b(G)$.

Definition 1.6. [4] For a graph *G* and a positive integer *b*, a set *S* of vertices in a graph is said to be a *b-disjunctive dominating set (b*DDS*)* if every vertex *v* not in *S* is either adjacent to a vertex of *S* or there are at least *b* vertices of *S* within distance *2* of *v* (or both). For a graph *G*, the minimum cardinality of a *b*-disjunctive dominating set is the *b-disjunctive domination number*, denoted by $\gamma_b^d(G)$.

Definition 1.7. [6] A set *S* of vertices in *G* is a *disjunctive total dominating set* of *G* if every vertex is adjacent to a vertex of *S* or has at least two vertices in *S* at distance 2 from it. The *disjunctive total domination number*, $\gamma_t^d(G)$, is the minimum cardinality of such a set.

Definition 1.8. [7] Let *G = (V,E)* be a connected graph with at least two vertices. For a fixed positive integer $b > 1$, a set $D \subseteq V$ is called a *b-disjunctive total dominating set* (*b*DTDS) of G if for every vertex $v \in V$, *v* is either adjacent to a vertex of *D* or has at least *b* vertices in *D* at distance 2 from it. The minimum cardinality of a *b*-disjunctive total dominating set of *G* is called the *b-disjunctive total domination number* of *G* and is denoted by $\gamma_b^{td}(G)$.

II. Main Results.

First we discuss about the relationships of various disjunctive domination number of a graph.

Theorem 2.1. Every 2-dominating set is a *b*-disjunctive dominating set.

Proof. Let *S* be a 2-dominating set. Then every vertex *v* not in *S* has at least 2 neighbours in *S*. That is, every vertex *v* not in *S* is adjacent to a vertex of *S*. Therefore, S is a *b*-disjunctive dominating set.

Theorem 2.2. For any graph *G*, $\gamma_2^d(G) \leq \gamma_t^d(G)$.

Proof. Let S be a disjunctive total dominating set of G. Then every vertex is adjacent to a vertex of S or has at least two vertices in S at distance 2 from it. Then every vertex v not in S is either adjacent to a vertex of S or there are at least 2 vertices of S within distance 2 of v. Therefore, S is a **2**-disjunctive dominating set.

Observation 2.3. For any graph G, the following hold.
(i). When b = 2, $\gamma_t^d(G) = \gamma_b^{td}(G)$ and
(ii). $\gamma_b^{td}(G) \le \gamma_{b+1}^{td}(G)$. In particular, $\gamma_t^d(G) \le \gamma_b^{td}(G)$ for any b.

Lemma 2.4. If v is a support vertex of a graph G with exactly one neighbour w that is not a leaf, then there is a $\gamma_b^{td}(G)$-set that contains v. Also, if $d(w) = 2$, then there is a $\gamma_b^{td}(G)$ - set which contains both v and w where v is a support vertex.
Proof. Let S be a $\gamma_b^{td}(G)$-set. Since v is a support vertex in a graph G with exactly one neighbour w that is not a leaf, to b-disjunctively dominate the leaf neighbours of v, at least b leaf neighbours of v belong to S. But to get the b-disjunctive dominating set with minimum cardinality, we can replace all the leaf neighbours of v in S with the vertex v. Therefore, $v \in S$. Further if $d(w) = 2$ and $w \notin S$, then at least one leaf neighbor of v belongs to S in order to totally dominate or disjunctively totally dominate v. We can replace such a leaf neighbor of v in S with the vertex w.

Lemma 2.5. [4] For b ≥ 3, $\gamma_b^d(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

The following Theorem 2.6 describes the realization theorem of b-disjunctive domination number and domination number.

Theorem 2.6. For any two positive integers b and y, if $b = y$, then there exists a connected graph G with $\gamma_b^d(G) = \gamma(G) = b$ and if $b < y$, then there exists a connected graph G with $\gamma_b^d(G) = \gamma_t^d(G) = b$ and $\gamma(G) = y, b \ge 2$.
Proof. For $b = y = k \ge 1$, let G be the cycle of 3k vertices. Then by Lemma 2.5, $\gamma_b^d(G) = \gamma(G) = \frac{3k}{3} = 3$. For $b < y$, let G be the graph of order 2y obtained from the Corona product of K_y and K_1. Every vertex of K_y form the minimal dominating set with minimum cardinality. Therefore, $\gamma(G) = y$. Since $d(u,v) \le 2$ for all $u \in V(K_y)$ and $v \in V(G) - V(K_y)$, for $2 \le b < y$, $\gamma_b^d(G) = b = \gamma_t^d(G)$.

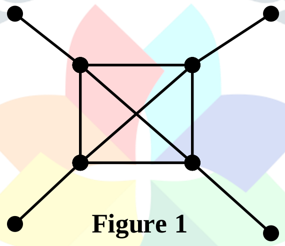

Figure 1

Notation 2.7. [4] Let v be the vertex of a graph. Then $N_1(v)$ is the set of vertices which are at distance one from v and $N_2(v)$ is the set of vertices which are at distance two from v.

The following Theorem 2.8 determines the disjunctive total domination number of $P_2 \square C_n$.

Theorem 2.8. Let $P_2 \square C_n$, $n \ge 3$ be a graph. Then $\gamma_t^d(P_2 \square C_n) = \begin{cases} \left\lceil \frac{n}{4} \right\rceil + 1 & \text{if } n \equiv 0,3 (mod\ 4) \\ \left\lceil \frac{n}{4} \right\rceil & \text{otherwise} \end{cases}$.

Proof. Consider the Cartesian product $P_2 \square C_n$. Let $v_1, v_2, \dots, v_n$ and $u_1, u_2, \dots, u_n$ be the vertices of the outer and inner cycle in $P_2 \square C_n$. Then $|V(P_2 \square C_n)| = 2n$ and $|E(P_2 \square C_n)| = 3n$. For $n \ge 3$, let $S = \begin{cases} \{v_1, u_1, v_5, u_5, \dots, v_{4i+1}, u_{4i+1}\} \cup \{v_n, u_n\} & \text{if } n \equiv 0,3 (mod\ 4) \\ \{v_1, u_1, v_5, u_5, \dots, v_{4i+1}, u_{4i+1}\} & \text{otherwise} \end{cases}$. Since each vertex in $P_2 \square C_n$ is either in S or adjacent to at least one vertex in S or there are at least two vertices in S at distance two from it, S is a disjunctive total dominating set of $P_2 \square C_n$. Therefore, $\gamma_2^d(P_2 \square C_n) \le |S| = \begin{cases} \left\lceil \frac{n}{4} \right\rceil + 1 & \text{if } n \equiv 0,3 (mod\ 4) \\ \left\lceil \frac{n}{4} \right\rceil & \text{otherwise} \end{cases}$. Since any vertex removal from S affects the disjunctive total property, S is a minimum disjunctive dominating set.

The following Theorem 2.9 gives the lower bound for b-disjunctive total domination number of the graph $P_2 \square C_n$.

Theorem 2.9. Let $P_2 \square C_n$, $n \ge 3$ be a graph. Then for $b \ge 3$, $\gamma_b^{td}(P_2 \square C_n) \ge \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1 & \text{if } n \equiv 2 (mod\ 4) \\ \left\lceil \frac{n}{2} \right\rceil & \text{otherwise} \end{cases}$.

Proof. Consider the Cartesian product $P_2 \square C_n$. Let $v_1, v_2, \dots, v_n$ be the vertices of the outer cycle and $u_1, u_2, \dots, u_n$ be the vertices of the inner cycle in $P_2 \square C_n$. Then $|V(P_2 \square C_n)| = 2n$ and $|E(P_2 \square C_n)| = 3n$. For $b \ge 3$ and $n \ge 3$, let $S = \begin{cases} \{v_1, u_3, v_5, u_7, \dots, v_{4i+1}, u_{4i+3}\} & \text{if } n \equiv 0,3 (mod\ 4) \\ \{u_3, v_5, u_7, v_9 \dots, v_{4j-1}, u_{4j+1}\} \cup \{v_1, v_2\} & \text{if } n \equiv 2 (mod\ 4) \\ \{u_3, v_5, u_7, v_9 \dots, v_{4k-1}, u_{4k+1}\} \cup \{v_1\} & \text{if } n \equiv 1 (mod\ 4) \end{cases}$ where $0 \le i \le \left\lceil \frac{n-1}{4} \right\rceil$, $1 \le j \le \left\lceil \frac{n-3}{4} \right\rceil$ and $1 \le k \le \left\lceil \frac{n-1}{4} \right\rceil$ with

$|S| = \left\lceil\frac{n}{2}\right\rceil + 1$ for $n \equiv 2(mod\ 4)$ and $|S| = \left\lceil\frac{n}{2}\right\rceil$ for otherwise. Each vertex in V – S is adjacent to at least one vertex in S or there are at least b vertices in S at distance two from it. We enlarge S to bDTDS by adding some vertices of V –S. Therefore, $\gamma_b^d(P_2 \square C_n) \geq |S| = \begin{cases} \left\lceil\frac{n}{2}\right\rceil + 1 & \text{if } n \equiv 2(mod\ 4) \\ \left\lceil\frac{n}{2}\right\rceil & \text{otherwise} \end{cases}$.

Next, we discuss about the b-disjunctive total domination number of the graph hypercube Q_n. For that, we have the following observations.

Observation. 2.10 (i) . For Q_1 and Q_2, $\gamma_t^d(Q_1) = 2$ and
(ii). For Q_3 and Q_4, $\gamma_t^d(Q_2) = 4$.

The following Theorem 2.11 determines the disjunctive total domination number of the hypercube.

Theorem 2.11. Let Q_n be the hypercube. Then for $5 \leq n \leq 8$, $\gamma_t^d(Q_n) = 2^{n-3}$.
Proof. Consider the hypercube Q_n. Let $v_1^i, v_2^i, \ldots, v_8^i$ be the vertices of the cube Q_4 which are in the 1st column and in the ith row of Q_n and $u_1^i, u_2^i, \ldots, u_8^i$ be the vertices of the cube Q_4 in the 2nd column and in the ith row of Q_n where $1 \leq i \leq 2^{n-4}$. Then $|V(Q_n)| = 2^n$ and $|E(Q_n)| = n\ 2^{n-1}$. For $5 \leq n \leq 8$, let $S = \{v_1^1, v_1^2, \ldots, v_1^{2^{n-4}}\} \cup \{u_7^1, u_7^2, \ldots, u_7^{2^{n-4}}\}$ with $|S| = 2^{n-3}$. Since each vertex in Q_n is either in S or adjacent to a vertex in S or there are at least *2* vertices of S within distance two from it, S is a disjunctive total dominating set of Q_n. Therefore, $\gamma_t^d(Q_n) \leq 2^{n-3}$. Let $W = \{w_1, w_2, \ldots, w_k\}$ be a minimum disjunctive total dominating set of Q_n. Let $v \in W$. Each vertex in Q_n can dominate itself and *n* distinct vertices in $N_1(v)$ and contribute $\frac{1}{2}$ to $n(n-1)$ distinct vertices in $N_2(v)$. Also, each vertex in W can dominate itself and five distinct vertices in $N_1(v)$ and contribute $\frac{1}{2}$ to six distinct vertices in $N_2(v)$. Since $|W| = k$, we get $k = 2^{n-3}$. That is, $|W| = k = 2^{n-3} = |S|$. Therefore, S is a minimum disjunctive total dominating set of Q_n.

The following Theorem 2.12 and 2.13 establish the b-disjunctive total domination number of Q_n if the value of n is between 5 and 8 and $b \geq 3$.

Theorem 2.12. Let Q_n be the hypercube. Then for $5 \leq n \leq 8$, $\gamma_3^{td}(Q_n) = 3(2^{n-4})$.
Proof. Consider the hypercube graph Q_n. Let $v_1^i, v_2^i, \ldots, v_8^i$ be the vertices of the cube Q_4 which are in the 1st column and in the ith row of Q_n and $u_1^i, u_2^i, \ldots, u_8^i$ be the vertices of the cube Q_4 in the 2nd column and in the ith row of Q_n where $1 \leq i \leq 2^{n-4}$. Then $|V(Q_n)| = 2^n$ and $|E(Q_n)| = n\ 2^{n-1}$. For disjunctive total dominating set and for $5 \leq n \leq 8$, let $S = \{v_1^1, v_1^2, \ldots, v_1^{2^{n-4}}\} \cup \{u_1^1, u_1^2, \ldots, u_1^{2^{n-4}}\} \cup \{v_7^1, v_7^3, \ldots, v_7^{2^{n-4}-1}\} \cup \{u_7^2, u_7^4, \ldots, u_7^{2^{n-4}}\}$ with $|S| = 3(2^{n-4})$. Since each vertex in Q_n is either in S or adjacent to a vertex in S or there are at least 3 vertices of S within distance two from it. Therefore, $\gamma_3^{td}(Q_n) \leq 3(2^{n-4})$. Let $W = \{w_1, w_2, \ldots, w_k\}$ be a minimum 3DTDS of Q_n. Let $v \in W$. Each vertex in Q_n can dominate itself and n distinct vertices in $N_1(v)$ and contribute $\frac{1}{3}$ to $n(n-1)$ distinct vertices in $N_2(v)$. Also, $(k - 2^{n-4})$ distinct vertices in W can dominate itself and three distinct vertices in $N_1(v)$ and 2^{n-4} distinct vertices in W can dominate itself and four distinct vertices in $N_1(v)$. Also, $(k - 2(2^{n-4}))$ distinct vertices in W can contribute $\frac{1}{3}$ to no vertex in $N_2(v)$, one of 2^{n-4} distinct vertices in W can contribute $\frac{1}{3}$ to three distinct vertices in $N_2(v)$ and another one of 2^{n-4} distinct vertices in W can contribute $\frac{1}{3}$ to six distinct vertices in $N_2(v)$. Since $|W| = k$, we get $4(k - 2^{n-4}) + 5(2^{n-4}) + 0(k - 2(2^{n-4})) + 1(2^{n-4}) + 2(2^{n-4}) = 2^n$. This implies $4k + (2^{n-4}) + 3(2^{n-4}) = 2^n$ and hence $k = 3(2^{n-4})$. That is, $|W| = k = 3(2^{n-4}) = |S|$. Therefore, S is a minimum 3DTDS of Q_n. Hence $\gamma_3^{td}(Q_n) = 3(2^{n-4})$.

Theorem 2.13. Let Q_n be the hypercube. Then for $5 \leq n \leq 8$ and $b \geq 4$, $\gamma_b^{td}(Q_n) = 2^{n-2}$.
Proof. Consider the hypercube graph Q_n. Let $v_1^i, v_2^i, \ldots, v_8^i$ be the vertices of the cube Q_4 which are in the 1st column and in the ith row of Q_n and $u_1^i, u_2^i, \ldots, u_8^i$ be the vertices of the cube Q_4 in the 2nd column and in the ith row of Q_n where $1 \leq i \leq 2^{n-4}$. Then $|V(Q_n)| = 2^n$ and $|E(Q_n)| = n\ 2^{n-1}$. For $5 \leq n \leq 8$, we construct the vertex set S of Q_n as follows. $S = \{v_1^1, v_1^2, \ldots, v_1^{2^{n-4}}\} \cup \{u_1^1, u_1^2, \ldots, u_1^{2^{n-4}}\} \cup \{v_7^1, v_7^2, \ldots, v_7^{2^{n-4}}\} \cup \{u_7^1, u_7^2, \ldots, u_7^{2^{n-4}}\}$ with $|S| = 2^{n-2}$. Since each vertex in Q_n is either in S or adjacent to a vertex in S or there are at least b vertices of S within distance two from it. Therefore, S is a b-disjunctive total dominating set of Q_n. Hence $\gamma_b^{td}(Q_n) \leq |S| = 2^{n-2}$. Let $W = \{w_1, w_2, \ldots, w_k\}$ be a minimum bDTDS and dominating set of Q_n. Let $v \in W$. Each vertex in Q_n can dominate itself and n distinct vertices in $N_1(v)$ and contribute $\frac{1}{b}$ to no vertex in $N_2(v)$ where $b \geq 4$. Also each vertex in W can dominate itself and three distinct vertices in $N_1(v)$ and contribute $\frac{1}{b}$ to no vertex in $N_2(v)$. Since $|W| = k$, we get $k + 3k = 2^n$ which implies $k = 2^{n-2}$. That is, $|W| = k = 2^{n-2} = |S|$. Therefore, S is a minimum bDTDS and dominating set of Q_n. Thus, for $4 \leq n \leq 8$ and $b \geq 4$, $\gamma_b^{td}(Q_n) = \gamma(Q_n) = 2^{n-2}$.

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