Generalization of Some Fixed Point Theorems in POSET related Metric Space

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Abstract: In present work, we introduces the new generalizations of Banach Fixed Point Theorem and Hardy-Roger Fixed Point Theorem and its consequence as Kannan Fixed Point Theorem in the POSET related metric space.

Index Terms: Fixed Point, Contraction Mappings, POSET.

1. INTRODUCTION

Let (X, d) be a metric space and a mapping T: X → X is said to be a contraction mapping if there exists $\lambda \in [0,1)$ such that for all x, y ∈ X we have d(Tx, Ty) ≤ λ d(x, y)

In 1922 Banach [1] proved that this contraction mapping has a unique fixed point Tx = x in a complete metric space (X, d).

In 2004 Ran and Reurings [2] introduced the Banach Fixed Point Theorem in ordered metric space as follows

Theorem 1.1

Let (X, ≼) be a partially ordered set with a metric d, then (X, d) be a complete metric space. Also, every pair x, y ∈ X has a lower bound and an upper bound. If f is a continuous, monotone self-map from X into X then there exists $\lambda \in (0,1)$ such that d(fx, fy) ≤ λ d(x, y) , x ≥ y and there exists x_0 ∈ X such that $x_0 \leq fx_0$ or $x_0 \geq fx_0$ then f has a unique fixed point $\hat{x}$. Moreover, for every x∈ X, $\lim_{n\to\infty} f^n x = \hat{x}$.

Also there are some following different Fixed Point results related to different contraction maps and some basic definitions are as follows.

Theorem 1.2

Let (X, d) be a metric space and T is a self-map on X satisfying the condition for x, y ∈ X,

d (Tx, Ty) ≤ λ_1d (x , y) + λ_2 d (x , Tx) + λ_3 d (y, Ty) + λ_4 d (x, Ty) + λ_5 d (y, Tx)

for all x, y ∈ X, where $\lambda_i \geq 0$, for i = 1, 2, 3, 4, 5, such that $\lambda = \sum_{i=1}^{5} \lambda_i$ and $\lambda \in [0, 1)$.

This contraction is called as Hardy-Roger Contraction. [4] Also, if

(I) X is Complete and λ <1, T has a unique fixed point.

(II) If x ≠ y and Hardy-Roger Contraction implies X is compact and T is continuous with λ = 1 then T has a unique fixed point.

Theorem 1.3

Let (X, d) be a complete metric space and a self-mapping T: X → X defined as

d (Tx, Ty) ≤ α[d (x, Tx) + d (y, Ty)] where $\alpha \in \left[0, \frac{1}{2}\right[$ and for all x, y ∈ X,

then T has a unique fixed point. Then this mapping is called Kannan type mapping and result is known as Kannan Fixed Point Theorem [3].

Definition 1.4 [4]

Let (X, d) be a metric space and T is a self-map on X is said to be sequentially convergent for every sequence $\{x_n\}$, if $\{Tx_n\}$ is convergent then $\{x_n\}$ is also convergent.

Definition 1.5

Let S be the set of all functionsΨ : [0, ∞) → [0, ∞) satisfying the following conditions

(I) Ψ is continuous and monotonic increasing

(II) Ψ (x) = 0 iff x = 0

Remark: Now onwards from here wereferred POSET as partially ordered set in this paper.

Theorem 1.6

Let (X, ≼) be a POSET with a metric d and (X, d) be a complete metric space. Let f: X → X be a monotonic increasing self- map and T is one to one, continuous, subsequentially convergent order preserving or increasing self-map with Ψ ∈ S.

Then for all x, y ∈ X with x ≼ y, λ ∈ [0, 1) andΨ (d (Tfx, Tfy)) ≤ λ Ψ (d(Tx, Ty))

Also, suppose that either

(I) f is continuous or

(II) Assume that if any increasing sequence $\{x_n\}$ in X converges to z, then x_n ≼ z for all n ≥ 0.

If there exists x_0 ∈ X with x_0 ≼ fx_0 , then f has a fixed point in X. Moreover, if for each x, y ∈ X there exists z ∈ X which is comparable to x and y , then the fixed point is unique.

Proof: Let x_0 ∈ X be an arbitrary point such that $x_n = f^n x_0$, for all n ∈ N .As f is monotonic increasing and x_0 ≼ fx_0

and $Tx_n \preccurlyeq Tx_{n+1}$ we have $Tx_0 \preccurlyeq Tfx_0 \preccurlyeq Tf^2x_0 \preccurlyeq \dots \preccurlyeq Tf^nx_0 \preccurlyeq \dots$

Ψ (d(Tx_n, Tx_{n+1}))= Ψ (d(Tfx_{n-1}, Tfx_n)) $\le \lambda\Psi$ (d(Tx_{n-1}, Tx_n))

$\le \lambda\Psi$ (d(Tfx_{n-2}, Tfx_{n-1}))

$\le \lambda^2\Psi$ (d(Tx_{n-2}, Tx_{n-1}))

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Continuing the process n times we get

Ψ (d (Tx_n, Tx_{n+1})) $\le \lambda^n\Psi$ (d(Tx_0, Tx_1)) , as n → ∞ and $\lambda \in [0, 1)$

We have

Ψ (d (Tx_n, Tx_{n+1})) → 0 since d(Tx_n, Tx_{n+1})→ 0

Also, for m, n ∈ N , m > n then Ψ (d (Tx_n, Tx_m)) $\le \lambda^n\Psi$ (d(Tx_0, Tx_{m-n}))

Let m, n → ∞ then we get d(Tx_n, Tx_m)→ 0 hence Ψ (d (Tx_n, Tx_m)) → 0.

Hence, we get a sequence $\{Tx_n\}$ which is Cauchy sequence but (X, d) be a complete metric space then there exists ω ∈ X such that $\lim_{n\to\infty} Tx_n = \omega$.

As T is one to one, continuous, subsequentially convergent order preserving self-map on X, so that the sequence $\{x_n\}$ has a convergent subsequence then there exists $\widehat{\omega}$ ∈ X such that $\lim_{k\to\infty} Tx_{n_k} = \widehat{\omega}$.

As T is continuous and $x_{n_k} \to \widehat{\omega}$ therefore $\lim_{k\to\infty} Tx_{n_k} = T\widehat{\omega}$ and $\lim_{k\to\infty} d(Tx_{n_k}, T\widehat{\omega}) = 0$.

Now we prove $\widehat{\omega}$ ∈ X is a fixed point of f in two cases as follows:

Case(I):- Suppose that f is continuous thenby continuity of f, we have

$T\widehat{\omega} = \lim_{k\to\infty} Tx_{n_k} = \lim_{k\to\infty} Tx_{n_{k-1}} = Tf\widehat{\omega}$

But T is one to one we get $f\widehat{\omega} = \widehat{\omega}$, this shows that $\widehat{\omega}$ is a fixed point of f.

Case(II):- Suppose that if any increasing sequence $\{x_n\}$ in X converges to z, then $x_n \preccurlyeq$ z for all n ≥ 0.

As $\{Tx_{n_k}\}$ converges to $T\widehat{\omega}$ ∈ X for all ϵ> 0 there is N_1 ∈ N such that for all n_k> N we have d(Tx_{n_k} , $T\widehat{\omega}$) <ϵ this gives us $Tx_{n_k} \le T\widehat{\omega}$ andΨ (d($Tf^{n_{k+1}}x$, $Tf\widehat{\omega}$)) $\le \lambda$ Ψ (d($Tf^{n_{k+1}}x$, $Tf\widehat{\omega}$)) as k → ∞ we get $T\widehat{\omega} = Tf\widehat{\omega}$ but T is one to one we get $\widehat{\omega}$ ∈ X as a fixed pointof f.

Now we prove the uniqueness of the fixed point by showing $\lim_{k\to\infty} f^{n_k}x = \widehat{\omega}$ for every x ∈ X.

Let x and x_0 be comparable then $x \preccurlyeq x_0$ implies $f^{n_k}x \preccurlyeq f^{n_k}x_0$ or $x_0 \preccurlyeq x$ implies $f^{n_k}x_0 \preccurlyeq f^{n_k}x$

Hence we get $\lim_{k\to\infty} f^{n_k}x = \lim_{k\to\infty} f^{n_k}x_0 = \widehat{\omega}$.

If x and x_0 are not comparable then x_1 , x_2 are upper bound, lower bound of x and x_0 respectively, then $x_2 \preccurlyeq x \preccurlyeq x_1$ and $x_2 \preccurlyeq x_0 \preccurlyeq x_1$ gives us $\lim_{k\to\infty} f^{n_k}x = \lim_{k\to\infty} f^{n_k}x_0 = \widehat{\omega}$.

Hence fixed point is unique.

2.GENERALIZATION OF FIXED POINT THEOREMS

Theorem 2.1

Let (X, ≼) be a POSET with a metric d and (X, d) be a complete metric space. Let f: X → X be a monotonic increasing self-map and T is one to one, continuous, subsequentially convergent order preserving or increasing self map with Ψ ∈ S.

Then for all x, y ∈X with x ≼ y, λ_i ∈ [0, 1) and

Ψ(d(Tfx, Tfy))≤ $\lambda_1\Psi$ (d(Tx , Ty)) + $\lambda_2\Psi$(d(Tx ,Tfx)) + $\lambda_3\Psi$(d(Ty, Tfy))+ $\lambda_4\Psi$ (d(Tx, Tfy)) + $\lambda_5\Psi$ (d(Ty, Tfx))

where$\lambda_i \ge 0$, for i = 1, 2, 3, 4, 5, such that $\lambda = \sum_{i=1}^{5}\lambda_i$

Also, suppose that either

(I) f is continuous or

(II) Assume that if any increasing sequence $\{x_n\}$ in X converges to z, then $x_n \preccurlyeq$ z for all n ≥ 0.

If there exists x_0 ∈ X with $x_0 \preccurlyeq fx_0$, then f has a fixed point in X. Moreover, if for each x, y ∈ X there exists z ∈ X which is comparable to x and y , then the fixed point is unique.

Proof:-Let x_0 ∈ X be an arbitrary point such that $x_n = f^nx_0$, for all n ∈ N .As f is monotonic increasing and $x_0 \preccurlyeq fx_0$ and $Tx_n \preccurlyeq Tx_{n+1}$ we have $Tx_0 \preccurlyeq Tfx_0 \preccurlyeq Tf^2x_0 \preccurlyeq \dots \preccurlyeq Tf^nx_0 \preccurlyeq \dots$

Ψ (d (Tx_n, Tx_{n+1})) = Ψ (d (Tfx_{n-1}, Tfx_n))

$\le \lambda_1\Psi$ (d (Tx_{n-1}, Tx_n)) + $\lambda_2\Psi$ (d (Tx_{n-1}, Tfx_{n-1})) + $\lambda_3\Psi$ (d (Tx_n, Tfx_n)) + $\lambda_4\Psi$ (d (Tx_{n-1} ,Tfx_n)) + $\lambda_5\Psi$ (d (Tx_n, Tfx_{n-1}))

$\le \lambda_1\Psi$ (d (Tx_{n-1}, Tx_n)) + $\lambda_2\Psi$ (d (Tx_{n-1}, Tx_n)) + $\lambda_3\Psi$ (d (Tx_n, Tx_{n+1})) + $\lambda_4\Psi$ (d (Tx_{n-1} ,Tx_{n+1})) + $\lambda_5\Psi$ (d (Tx_n, Tx_n))

$\le (\lambda_1 + \lambda_2)\Psi$ (d (Tx_{n-1}, Tx_n)) + $\lambda_3\Psi$ (d (Tx_n, Tx_{n+1})) + $\lambda_4\Psi$ [d (Tx_{n-1} ,Tx_n) + d (Tx_n ,Tx_{n+1})]

$\le (\lambda_1 + \lambda_2 + \lambda_4)\Psi$ (d (Tx_{n-1}, Tx_n)) + $(\lambda_3 + \lambda_4)\Psi$ (d (Tx_n, Tx_{n+1}))

Hence we get, Ψ (d (Tx_n, Tx_{n+1}))≤ $\left(\frac{\lambda_1+\lambda_2+\lambda_4}{1-\lambda_3-\lambda_4}\right)\Psi$ (d (Tfx_{n-1}, Tfx_n))

Continuing the process n times we get

Ψ (d $(Tx_n$, Tx_{n+1}))$\leq \left(\frac{\lambda_1+\lambda_2+\lambda_4}{1-\lambda_3-\lambda_4}\right)^n \Psi$ (d $(Tx_0$, Tx_1))
as n $\to \infty$ we get Ψ (d $(Tx_n$, Tx_{n+1})) $\to 0$.
For all m, n $\in$ N , taking m > n, we have Ψ (d $(Tx_n$, Tx_m)) = Ψ (d $(Tf^n x_0$, T$f^m x_0$))
$\leq \left(\frac{\lambda_1+\lambda_2+\lambda_4}{1-\lambda_3-\lambda_4}\right)^n \Psi$ (d $(Tx_0$, Tx_1))
as m, n $\to \infty$ we get Ψ (d $(Tx_n$, Tx_m)) $\to 0$.
So we have d $(Tx_n$, Tx_m) $\to 0$ as m, n $\to \infty$.
Hence, we get a sequence $\{Tx_n\}$ which is a Cauchy sequence in a complete metric space (X, d) and then there exists $\omega \in$ X such that converges to T$\omega \in$ X .
Now, we prove$\omega \in$ X is a fixed point of f in two cases; as similar approach as given in Theorem 2.1.

Case(I):- Suppose that f is continuous thenby continuity of f, we have
$T\widehat{\omega} = \lim_{k\to\infty} Tx_{n_k} = \lim_{k\to\infty} Tx_{n_{k-1}} = Tf\widehat{\omega}$
But T is one to one we get $f\widehat{\omega} = \widehat{\omega}$, this shows that $\widehat{\omega}$ is a fixed point of f.

Case(II):- Suppose that if any increasing sequence $\{x_n\}$ in X converges to z, then $x_n \preccurlyeq$ z for all n ≥ 0. As $\{Tx_n\}$ converges to $T\omega \in$ X for all $\epsilon > 0$ there is $N_1 \in$ N such that for all n >N_1 we get d $(Tx_n$, Tω) <ϵ.
Also, as $\{Tx_n\}$ converges to $T\omega$ we get $Tx_n \preccurlyeq T\omega$ and
Ψ (d $(T\omega, Tf\omega)) \leq \Psi$ [(d$(T\omega, Tx_n)) + \Psi$ (d$(Tx_n, T\omega)$)]

$\leq \Psi$[λ_1 (d $(T\omega, Tx_n)$ + λ_2 d (Tx_n, Tx_{n-1}) + λ_3 (d ($T\omega$, T$f\omega$) + λ_4 d (Tx_{n-1} ,Tfx_n) + λ_5 (d $(Tfx_{n-1}$, T$f\omega$)]

Ψ (d $(T\omega, Tf\omega)) \to 0$, as n $\to \infty$ this implies $T\omega = Tf\omega$ but T is one to one we have $\omega \in$ X is a fixed point of f.

Uniqueness of the fixed point follows from Hardy-Roger Contraction.

Theorem 2.2
Let (X, $\preccurlyeq$) be a POSET with a metric d and (X, d) be a complete metric space. Let f: X $\to$ X be a monotonic increasing self- map and T is one to one, continuous, subsequentially convergent order preserving or increasing self-map with $\Psi \in$ S.
For all x, y $\in$X with x $\preccurlyeq$ y, $\alpha \in \left[0, \frac{1}{2}\right[$ andΨ (d(Tfx, Tfy)) $\leq \alpha$[Ψ (d(Tx, Tfx)) + Ψ (d(Ty, Tfy))]
Also, suppose that either
(I) f is continuous or
(II) Assume that if any increasing sequence $\{x_n\}$ in X converges to z, then $x_n \preccurlyeq$ z for all n ≥ 0.

If there exists $x_0 \in$ X with $x_0 \preccurlyeq$ fx_0 , then f has a fixed point in X. Moreover, if for each x, y $\in X$ there exists z $\in$ X which is comparable to x and y , then the fixed point is unique.

Proof:-This is Kannan Fixed Point Theorem in POSET metric space and proof follows if we consider$\lambda_2 = \lambda_3 = \alpha$ and $\lambda_1 = \lambda_4 = \lambda_5 = 0$ in Theorem 2.1

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