Alternative Way of Shifting Guests in Hilbert's Hotel

Sukhwinder Singh

Department of Mathematics, J.C. D.A.V. College, Dasuya, India.

Abstract. Hilbert's grand hotel demonstrates the meaning of words like "full" or "exhaustive" in case of infinite sets. Hilbert used it as an example to show that infinity does not behave in the same way as regular numbers do. This paradox helps to explain the most important property of infinite sets i.e. an infinite set is equivalent to one of its proper subsets. In addition to the explanation of Hilbert's paradox of grand hotel, this paper presents an alternative way of shifting the guests when infinite number of new guests arrive in a completely filled hotel having infinite number of rooms.

Keywords: infinity, Hilbert's grand hotel, paradox.

Introduction

Infinity has very important place in Mathematics. While studying the properties of countable and uncountable sets, one must have proper knowledge of infinity. In order to explain the extraordinary behavior of infinity, Hilbert's Paradox was proposed by Hilbert.

Hilbert's Paradox is a story of an imaginary hotel having infinite number of rooms, numbered 1, 2, 3 … and each room occupied by a guest. Suppose a new guest arrives and asks for a room. Hilbert proposed that guest in room number 1 can be shifted to room number 2, guest in room number 2 can be shifted to room number 3 and so on. With this shifting, Room number 1 is vacant and can be given to the new guest. With the help of this trick, any finite number of new guests can be accommodated. If n guests arrive, ask each existing guest to move to the room whose number is n plus the number of their previously occupied room.

Now, suppose an infinite number of new guests arrive in the hotel and they are asking for rooms. In this case, ask each existing guest to move to the room whose number is twice the number of their current room. In this way, guest in room number 1 is shifted to room number 2, guest in room number 2 is shifted to room number 4 and so on… .For example, guest in room number 50 will be shifted to room number 100. After this shifting, odd numbered rooms are vacant and new guests can be accommodated in these rooms. From this situation, it is clear that set of odd numbers is equivalent to set of natural numbers. Similarly, we can say that set of even numbers and set of natural numbers have the same potential or cardinality.

Main Result

I consider the case when guests present in room number 1 and 2 are important persons. These persons cannot be asked to shift to the new rooms on the arrival of new guests. Now we try to shift the guests already present in the hotel in such a way that the guests present in room number 1 and 2 are not disturbed. For this situation, we define the partition of set of natural numbers as follows:

Consider the following sets

$S_1 = \{1\} = \{x_{11}\}$

$S_2 = \{2, 3\} = \{x_{21}, x_{22}\}$

$S_3 = \{4, 5, 6\} = \{x_{31}, x_{32}, x_{33}\}$

$S_4 = \{7, 8, 9, 10\} = \{x_{41}, x_{42}, x_{43}, x_{44}\}$

…

…

$S_k = \{x_{k1}, x_{k2}, \ldots\ldots\ldots\ldots, x_{kk}\}$

and so on..

Clearly, these sets are pairwise disjoint and their union is N. So, these sets will form a partition of N.

Define a function $f : N \rightarrow N$ as $f(n) = x_{n1}$ for all natural numbers n.

Clearly, f(1)=1 , f(2) =2 , f(3) = 4 , f(4)= 7 , f(5) = 11, f(6)= 16 and so on.

Range of f is { 1, 2, 4, 7, 11, 16, 22,…}

With this shifting, guests in room number 1 and 2 will remain in their rooms while guest in room number 3 is shifted to room number 4, guest in room number 4 is shifted to room number 7 and so on...With this arrangement, rooms with numbers in the set {3,5,6,8,9,10,12,13,14,15,...} will be vacant and infinitely many new guests can be accommodated in these rooms.

Now we find the new room number for the guest present in room number n

Case 1: n is even

Clearly, number of members in first set=1

Number of members in second set=2

Number of members in third set=3

So, number of members in nth set=n

Now, first set has one member and $(n-1)^{th}$ set has n-1 members. So, number of members in these two sets=1+n-1=n

Similarly, number of members in second and $(n-2)^{th}$ set=n

Number of members in $\left(\frac{n}{2}-1\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ set = $\frac{n}{2}-1+\frac{n}{2}+1$ =n

Also, number of members in $\left(\frac{n}{2}\right)^{th}$ set= $\frac{n}{2}$

Therefore, number of members in first n-1 sets=n $\left(\frac{n}{2}-1\right)+\frac{n}{2}=\frac{n(n-1)}{2}$

So, n^{th} set starts with $\frac{n(n-1)}{2}+1$. We can say that $\frac{n(n-1)}{2}+1$ is the image of n under f. So, new room number for the guest present in room number n is $\frac{n(n-1)}{2}+1$

Case 2: n is odd

Clearly, number of members in first set=1

Number of members in second set=2

Number of members in third set=3

So, number of members in nth set=n

Now, first set has one member and $(n-1)^{th}$ set has n-1 members. So, number of members in these two sets=1+n-1=n

Similarly, number of members in second and $(n-2)^{th}$ set=n

Number of members in $\left(\frac{n-1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}\right)^{th}$ set=n

Therefore, number of members in first n-1 sets=$\frac{n(n-1)}{2}$.

So, n^{th} set starts with $\frac{n(n-1)}{2}+1$. We can say that $\frac{n(n-1)}{2}+1$ is the image of n under f. So, new room number for the guest present in room number n is $\frac{n(n-1)}{2}+1$.

From both the cases, it is clear that guest present in room number n will be shifted to room number $\frac{n(n-1)}{2}+1$ when infinite number of new guests arrive in the hotel.

Conclusion

Hilbert's Paradox of grand hotel plays an important role in understanding the nature of infinity. It gives a practical proof of the result which states that set of even numbers and set of natural numbers have the same cardinality (Aleph Naught). Also this paradox helps to explain the property of infinite sets (an infinite set is equivalent to one of its proper subsets) which differentiate it from finite sets. In this paper, we presented an alternative way to shift the guests while we consider that guests in rooms numbered 1 and 2 cannot be asked to shift from their positions.

References

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