# Parametric Estimation under Step-Stress Partially Accelerated Life Tests for Inverse Rayleigh Distribution based on Adaptive Type-I Progressively Hybrid Censoring

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***Abstract :*** In this article, the maximum likelihood estimators of Inverse Rayleigh distribution parameters and the acceleration factor are discussed based on two different types of progressively hybrid censoring schemes for step-stress partially accelerated life test. The performances of the estimators of the model parameters using these censoring schemes are evaluated and also compared in terms of biases and mean squared errors using the Monte Carlo simulation Technique.
**Keywords:** Step-stress, Accelerated Life Test, Inverse Rayleigh Distribution, Progressively hybrid censoring, Adaptive progressively hybrid censoring, a Simulation study.

## I. INTRODUCTION

It is a very tough task to gather lifetimes on highly trustworthy items/products that have very lengthy lifetimes since a small number of products may occur within a limited time of testing with normal operating circumstances in reliability investigation. Then, a test called accelerated life test (ALT), or partially accelerated life test (PALT) is introduced, which is frequently used to induce untimely failure. The test is called ALT if all test units are uncovered to higher than usual levels of stress, and some of the units are run under strict condition, then we can say that the test is PALT. In the accelerated or partially situation, the data obtained from the experiment is used to calculate the authentic performance of the item in the usual condition.

There are many ways of applying stress, but two methods are most common and suitable, i.e., constant stress and step-stress. According to Nelson [1], the test unit is first to run at normal condition; if it does not fail for a particular time, then it is run at the accelerated condition until the test/experiment ends in step-stress PALT (SSPALT) whether the test unit is run either normal or accelerated condition only in constant-stress PALT (CSPALT). In other words, each unit is run a constant level of stress until it fails or censors.

There are many censoring schemes, but two censoring schemes, i.e., Type-I and Type-II, are the most popular censoring schemes in reliability analysis. But there is a drawback of these censoring schemes, i.e., we cannot draw items from the live test at any point other than the terminal point. To remove this difficulty, a censoring scheme comes in nature called progressive Type-II censoring or progressive Type-II hybrid censoring. There is one another major drawback of these censoring is that we cannot fix time and failure in advance. To remove this difficulty, the hybrid censoring scheme was introduced, which is a mixture of these two censoring schemes. The hybrid censoring scheme is described as follows

If the lifetime of test units is independent and identically distributed (i.i.d.) and $'n'$ is the number of units, which put on the test. Assume that $x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$ be the ordered lifetime of these units, respectively. The test ends in two cases (i) if a pre-specified number $m\ (<n)$ out of $n$ having unsuccessful, (ii) if pre-specified time $t_0$ has been reached. We cannot replace failed units in the test.

The hybrid censoring scheme is initially introduced by Epstein [2] and referred this scheme as Type-I hybrid censoring scheme (Type-I HCS). He introduced the Type-I HCS and also considered the particular case when the lifetime is Exponentially distributed with mean lifetime $\theta$. Estimation method for this mean lifetime $\theta$ is considered by him and proposed a two-sided confidence interval for this lifetime with no formal proof of its construction. Later, the Type-I HCS is discussed for many other lifetime distributions such as Weibull, Log-normal, Generalized Exponential, and two-parameter Exponential. The Type-I HCS has two special cases, i.e., Type-I and Type-II right censoring schemes.

Under progressive Type-I hybrid censoring scheme (PHCS-T I), the testing time and number of failures out of $n$ units put under the testing by the experimenter and the test terminates at the time $T = \min(m, t_0^*)$. From the occurrence of the first failure, $R_1$ units from the remaining (surviving) $n-1$ units are removed at random from the test. When the failure of the second unit occurs, the experimenter arbitrarily excludes $R_2$ units from the remaining $n-2-R_1$ units. This procedure continues until $T = \min(m, t_0^*)$ it is attained, and all the surviving units $R_j^* = n - j - R_1 - R_2 - ... - R_j$ are excluded from the experiment. Where $j$ denotes the number of units which observed up to time $T = \min(m, t_0^*)$ and $R_1, R_2, ..., R_j$ are the pre-specified whole numbers. For an overview of PHCS-T I, we may refer Balakrishnan and Aggarwala [3] and Balakrishnan [4]. But there is a major drawback in progressive hybrid censoring scheme, i.e., the experimenter is not sure about the ending time of the experiment. To remove this difficulty, another censoring proposed by Lin and Huang [5], called adaptive Type-I progressive hybrid censoring scheme (AT: I-PHCS). Hence, at the pre-specified time $t_0^*$, the AT: I-PHCS guarantees the ending of the life testing experiment that results in greater efficient estimators in comparison with PHCS-T I.

There is much literature available on SSPALT with different types of censoring schemes such as Soliman et al.[6] designed SSPALT for Inverse Weibull distribution based on progressive Type-II censoring scheme. A. Ismail [7,8,9] designed SSPALT for Weibull, Generalized Exponential and Binomial distributions under Type-II, Type-II and progressive Type-II censoring schemes, respectively. M. M. Mohie El-Din et al. [10] provided an inference on SSPALT using Extention of Exponential distribution based on progressive Type-II censoring scheme. They analyzed a real data set to point up the proposed procedures in his study. Rehman et al. [11,12] presented the estimation of Mukherjee-Islam model parameters under Type-II censored sample and presented a statistical analysis for Type-I progressive hybrid censoring scheme based on Burr Type-XII distribution under SSPALT. A. Ismail [13,14] presented an inference on SSPALT for Weibull distribution based on an adaptive Type-I progressive hybrid censoring scheme and an adaptive Type-II progressive hybrid censoring scheme. Li ling et al. [15] presented a parametric inference on simple step-stress accelerated life test for Exponential distribution based on progressive Type-I hybrid censoring scheme. In order to diminish the lifetime and lessen the test cost, the progressive Type-I hybrid censoring scheme and statistical technique in synthetic accelerated stresses are presented by them. Xiaolin et al. [16] discussed an inference and optimal design on SSPALT for the hybrid system based on progressive Type-II hybrid censoring. Showkat et al. [17] presented a SSPALT plan for Rayleigh distribution under adaptive Type-II, progressively hybrid censoring scheme. Nassar et al. [18] designed SSPALT for Burr Type-XII distribution and estimated the maximum likelihood estimators of parameters of distribution and acceleration factor for adaptive Type-I and Type-II progressively hybrid censoring schemes. Jing et al. [19] presented a reliability study in SSPALT for a series system, when a lifetime of units follow Burr Type-XII distribution. Yimin and Xiaolin [20] derived a SSPALT based on progressive Type-I hybrid censoring scheme when a lifetime of test units follows the Pareto distribution. Ashour and Nassar [21] considered a computing risk model for adaptive Type-I progressive censored data when a lifetime of test units follows the Weibull distribution. Ullah et al. [22] designed ALT for Generalized Rayleigh distribution with complete data using geometric process.

The rest of the article has the following sections and organized as follows. The model description and test material are provided in section 2. The point estimation of model parameters and the acceleration factor is provided in section 3. The interval estimation of model parameters and the acceleration factor is provided in section 4. A simulation technique for the comparison of two censoring scheme is provided in section 5. A concluding remark is provided in section 5.

## II. Model Description and Test Method

The Inverse Rayleigh distribution (IRD) is an important lifetime distribution in survival analysis. This distribution is introduced by Trayer [23]. A random variable (r.v.) $X$ is said to have the Inverse Rayleigh distribution if the probability density function $(pdf)$ with shape parameter $p$ takes the following form;

$$f_x(x,p) = \frac{2p^2}{x^3} e^{-(p/x)^2} \qquad x \ge 0, p > 0 \qquad (1)$$

The cumulative distribution function $(cdf)$ of IRD takes the following form

$$F_x(x,p) = e^{-(p/x)^2} \qquad x \ge 0, p > 0 \qquad (2)$$

The Survival function of IRD takes the following form

$$S_x(x,p)=1-e^{-(p/x)^2} \quad (3)$$

Some important properties of IRD are discussed by Voda [24], he discussed some important properties of maximum likelihood estimator (MLE) of shape parameters $p$ and also discussed that this distribution could approximate the several lifetime distributions. The survival function of IRD is increasing then decreases than after some time it becomes stable. This property of survival function is noticed by Mukherjee and Saran [25]. Many important methods of estimation for this distribution are proposed by Gharraph [26] and Mukherjee and Maiti [27].

Under SSPALT, the $pdf$ is given as

$$f(x)=\begin{cases}0 & x\le 0\\ f_1(x)=f_x(x,p) & 0<x\le\tau\\ f_2(x) & x>\tau\end{cases} \quad (4)$$

$$f_2(x)=\frac{2\beta p^2}{(\tau+\beta(x-\tau))}e^{-(p/\tau+\beta(x-\tau))^2} \quad x\ge 0,\, p>0 \quad (5)$$

The above $pdf$ is obtained by using the variable-transformation proposed by DeGroot and Goal [28], and the technique is given in the following Equation6.

$$X=\begin{cases}T & T\le\tau\\ \tau+\beta^{-1}(T-\tau) & T>\tau\end{cases} \quad (6)$$

Where $T$ is the lifetime of an item on usual operating conditions. $\tau$ and $\beta(>1)$ are the stress change time and acceleration factor, respectively.

Under progressive Type-I hybrid censoring scheme (PHCS-T I), a life test is completed at random time $\min(X_{m:n:n},t_0^*)$ under this censoring scheme, where $0<t_0^*<\infty$ and $0<t_0^*<\infty$ they are fixed already. The ordered failure times resulting from the test are $x_{1:m:n}\le x_{2:m:n}\le\ldots\le x_{j:m:n}$. If the $mth$ progressively censored observed failure occurs before time $t_0^*$ $(i.e.\ x_{m:n:n}<t_0^*)$, then the test ends at the time $t_0(=x_{m;m:n})$; otherwise, the test will be ended at time $t_0^*$, $(x_{j:m:n}<t_0^*<x_{j+1:m:n})$ and all the remaining $(n-\sum_{i=1}^{j}R_i-j)$ surviving units are censored at time $t_0^*$. Here $j$ is the number of failures that occur before time, $\min(X_{m:n:n},t_0^*)$.

Let the normal stress level be $S_1$ and the stress level at time $\tau$ be $S_2$. The test units first subjected to $S_1$, and after this, the stress is increased to $S_2$ at time $\tau$ under SSPALT. Under both stress levels, let the pre-specified number of failures be $m$. and along with removals $(R_1,R_2,R_{3,\ldots,}R_\tau,\ldots,R_j)$, the termination time $t_0^*$ is fixed already. $R_i$ test units are taken out from the test at the $ith$ failure time $x_{i:m:n}$ and at the time $\tau$, $R_\tau(>0)$ test units would be taken out from the surviving ones, and this process would continue.

Under PHCS-T I, the observed data in SSPALT is given as follows

Situation I- $t_0^*>x_{i:m:n}$, then

$$\left.\begin{aligned}S_1&=(x_{i:m:n},R_1),(x_{2:m:n},R_2)\ldots,(x_{m_u:m:n},R_{m_u}),(\tau,R_\tau),\\ S_2&=(x_{m_u+1:m:n},R_{m_u+1}),(x_{m_u+2:m:n},R_{m_U+2})\ldots,(x_{m-1:m:n},R_{m-1}),(x_{m:m:n},R_j^*)\end{aligned}\right\} \quad (7)$$

Situation II- $t_0^*=x_{i:m:n}$, then

$$\left.\begin{aligned}S_1&=(x_{i:m:n},R_1),(x_{2:m:n},R_2)\ldots,(x_{m_u:m:n},R_{m_u}),(\tau,R_\tau),\\ S_2&=(x_{m_u+1:m:n},R_{m_u+1}),(x_{m_u+2:m:n},R_{m_U+2})\ldots,(x_{m-1:m:n},R_{m-1}),(x_{m:m:n}=t_0^*,R_j^*=R_m)\end{aligned}\right\} \quad (8)$$

Situation III- $t_0^*<x_{i:m:n}$, then

$$\left.\begin{aligned}S_1&=(x_{i:m:n},R_1),(x_{2:m:n},R_2)\ldots,(x_{m_u:m:n},R_{m_u}),(\tau,R_\tau),\\ S_2&=(x_{m_u+1:m:n},R_{m_u+1}),(x_{m_u+2:m:n},R_{m_U+2})\ldots,(x_{j:m:n},R_j),(t_0^*,R_j^*)\end{aligned}\right\} \quad (9)$$

In the above equations, $m_u$ denotes the number of failures at normal condition.

Under SSPALT, the total number of failure '$j$' and finally censored number '$R_j^*$' in each case are respectively given in the following Equation10.

$$\left.\begin{aligned} j = m \, and \, R_j^* &= n - m - \sum_{i=1}^{n} R_i, if\ \tau < x_{i:m:n} < t_0^* \\ j = m \, and \ R_j^* &= n - m - \sum_{i=1}^{n} R_i, if\ x_{i:m:n} = t_0^* \\ j < m \, and \ R_j^* &= n - m - \sum_{i=1}^{n} R_i\ , if x_{i:m:n} > t_0^* \end{aligned}\right\} \quad (10)$$

Under AT: I-PHCS, there are $n$ units or items which are put on the test and also suppose that the ordered lifetime of these test items is $x_{1:n} < x_{2:n} < ... < x_{n:n}$. The experiment will not pause under the case the $mth$ failure $t_{m:n:n}$ occurs before the time $t_0^*$ $(i.e. t_0^* > x_{m:n:n})$. The experiment will continue to identify failures up to time $t_0^*$ without any further removals of items. The experiment will end when the time $t_0^*$ is reached, and all the remaining units $R_j^{**} = n - m - \sum_{i=1}^{m} R_i$ are taken off.

So, the observed data under SSPALT based on AT: I-PHCS is given as follows

Situation I- $t_0^* > x_{i:m:n}$, then

$$\left.\begin{aligned} S_1 &= (x_{i:m:n}, R_1), (x_{2:m:n}, R_2)..., (x_{m_u:m:n}, R_{m_u}), (\tau, R_\tau), \\ S_2 &= (x_{m_u+1:\,m:n}, R_{m_u+1}), (x_{m_u+2:m:n}, R_{m_U+2})..., (x_{m-1:m:n}, R_{m-1}), \\ &(x_{m:m:n}, R_m), (x_{m+1:m:n}, 0),..., (x_{j:m:n}, 0), (t_0^*, R_j^{**}) \end{aligned}\right\} \quad (11)$$

Situation II- $t_0^* = x_{i::m:n}$, then

$$\left.\begin{aligned} S_1 &= (x_{i:m:n}, R_1), (x_{2:m:n}, R_2)..., (x_{m_u:m:n}, R_{m_u}), (\tau, R_\tau), \\ S_2 &= (x_{m_u+1:\,m:n}, R_{m_u+1}), (x_{m_u+2:m:n}, R_{m_U+2})..., (x_{m:m:n} = t_0^*, R_m = R_j^{**}) \end{aligned}\right\} \quad (12)$$

Situation III- $t_0^* < x_{i:m:n}$, then

$$\left.\begin{aligned} S_1 &= (x_{i:m:n}, R_1), (x_{2:m:n}, R_2)..., (x_{m_u:m:n}, R_{m_u}), (\tau, R_\tau), \\ S_2 &= (x_{m_u+1:\,m:n}, R_{m_u+1}), (x_{m_u+2:m:n}, R_{m_U+2})..., (x_{m:m:n}, R_j), (t_0^*, R_j^{**}) \end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned} j = m \, and \, R_j^{**} &= n - j - \sum_{i=1}^{n} R_i\ , if\ \tau < x_{i:m:n} < t_0^* \\ j = m \, and \ R_j^{**} &= n - m - \sum_{i=1}^{n} R_i, if\ x_{i:m:n} = t_0^* \\ j < m \, and \ R_j^* &= n - j - \sum_{i=1}^{n} R_i, if\ x_{i:m:n} > t_0^* \end{aligned}\right\} \quad (14)$$

Therefore, adaptive progressive Type-I hybrid censoring scheme is a novel scheme that which provides the freedom of no particular bound in order to change the value of time $t_0^*$ to reach the optimal testing time, increasing greater chance of observing many failures. This censoring scheme gives assurance to the conclusion of the experiment at a fixed time and also monitors and controls the total number of failures for the experiment thus, not allow it to be too distant from the optimal failure number.

## III. Inference under Progressively Hybrid Censoring Scheme

On the basis of progressive hybrid censoring (PHC) and adaptive progressive hybrid censoring (APHC), point estimation for the model parameters and acceleration factor are obtained in this section. Here we take likelihood function for situation III (i.e. $t_0^* < x_{i:m:n}$) mentioned in the previous section because the difference between two censoring scheme lies in this situation only.

**Estimation under PHC**

The likelihood function of Inverse Rayleigh distribution under SSPALT based on PHC for $jth$ ordered lifetime data is given as

$$L(p, \beta) \propto \prod_{i=1}^{m_u} f_1(x_{i:m:n}) [1 - F_1(x_{i:m:n})]^{R_i} [1 - F_1(\tau)]^{R_\tau} \prod_{i=m_u+1}^{m} f_1(x_{i:m:n}) [1 - F_1(x_{i:m:n})]^{R_i} [1 - F_1(t_0)]^{R_j^*} \quad (15)$$

Where, $R_j^* = (n - \sum_{i=1}^{n} R_i - m)$

$m = m_u + m_a$ , $m_u$ is the number of failed units at use conditions and $m_a$ is the number of failed units at accelerated conditions. $t_0$ is the time of occurrence of $mth$ failure.

The log-likelihood function of Inverse Rayleigh distribution under SSPALT based on PHC for $jth$ ordered lifetime data is given as

$$\ln L = \sum_{i=1}^{m_u} \ln \frac{2p^2}{x_i^3} - \sum_{i=1}^{m_u} \ln\left(\frac{p}{x_i}\right)^2 + \sum_{i=1}^{m_u} R_i \ln\left[1 - e^{(p/x_i)^2}\right] + R_\tau m_u \ln\left[1 - e^{(p/\tau)^2}\right] +$$
$$+ \sum_{i=m_u+1}^{m} \ln \frac{2\beta p^2}{(\tau + \beta(x_i - \tau))^3} - \sum_{i=m_u+1}^{m} (p/\tau + \beta(x_i - \tau))^2 + \sum_{i=m_u+1}^{m} R_i \ln\left[1 - e^{-(p/\tau+\beta(x_i-\tau))^2}\right] + m_a R_J^* \ln\left[1 - e^{-(p/\tau+\beta(t_0-\tau))^2}\right]$$

$$\ln L = \sum_{i=1}^{m_u} \ln \frac{2p^2}{x_i^3} - \sum_{i=1}^{m_u} \ln\left(\frac{p}{x_i}\right)^2 + \sum_{i=1}^{m_u} R_i \ln\left[1 - e^{(p/x_i)^2}\right] + R_\tau m_u \ln\left[1 - e^{(p/\tau)^2}\right] +$$
$$+ \sum_{i=m_u+1}^{m} \ln \frac{2\beta p^2}{\psi_i^3} - \sum_{i=m_u+1}^{m} (p/\psi_i)^2 + \sum_{i=m_u+1}^{m} R_i \ln\left[1 - e^{-(p/\psi_i)^2}\right] + m_a R_J^* \ln\left[1 - e^{-(p/\psi_t)^2}\right] \quad (16)$$

Where, $\ln L = \ln L(p, \beta)$, $\psi_i = \tau + \beta(x_i - \tau)$ $\psi_t = \tau + \beta(t_0 - \tau)$

Now differentiate the above log-likelihood Equation16 with respect to $p$ and $\beta$ equating both equations to zero.

$$\frac{\partial \ln L}{\partial p} = \sum_{i=1}^{m_u} \frac{8p^3}{x_i^6} - \sum_{i=1}^{m_u} \frac{2p^3}{x_i^4} + \sum_{i=1}^{m_u} R_i e^{-(p/x_i)^2} \frac{(2p/x_i)^2}{1 - e^{-(p/x_i)^2}} + R_\tau m_u e^{-(p/\tau)^2} \frac{2p/\tau^2}{1 - e^{-(p/\tau)^2}} +$$
$$+ \sum_{i=m_u+1}^{m} \frac{8\beta^3 p^2}{\psi_i^6} - \sum_{i=m_u+1}^{m} \frac{2p}{\psi_i^2} + \sum_{i=m_u+1}^{m} \frac{e^{-(p/\psi_i)^2}\, 2p/\psi_i^2}{\left[1 - e^{-(p/\psi_i)^2}\right]} R_i + m_a R_J^* \frac{e^{-(p/\psi_t)^2}\, 2p/\psi_t^2}{\left[1 - e^{-(p/\psi_i)^2}\right]} = 0 \quad (17)$$

$$\frac{\partial \ln L}{\partial p} = \sum_{i=1}^{m_u} \frac{8p^3}{x_i^6} - \sum_{i=1}^{m_u} \frac{2p^3}{x_i^4} + \sum_{i=1}^{m_u} R_i e^{-(p/x_i)^2} \frac{(2p/x_i)^2}{1 - e^{-(p/x_i)^2}} + R_\tau m_u e^{-(p/\tau)^2} \frac{2p/\tau^2}{1 - e^{-(p/\tau)^2}} +$$
$$+ \sum_{i=m_u+1}^{m} \frac{8\beta^3 p^2}{\psi_i^6} - \sum_{i=m_u+1}^{m} \frac{2p}{\psi_i^2} + \sum_{i=m_u+1}^{m} \frac{e^{-(p/\psi_i)^2}\, 2p/\psi_i^2}{\left[1 - e^{-(p/\psi_i)^2}\right]} R_i + m_a R_J^* \frac{e^{-(p/\psi_t)^2}\, 2p/\psi_t^2}{\left[1 - e^{-(p/\psi_i)^2}\right]} = 0 \quad (18)$$

From the above Equation17 and 18, it looks impossible to get a closed-form solution to these equations. So, Newton –Raphson technique is applied to get the ML estimates of $p$ and $\beta$ by using these equations.

**Estimation under APHC**

The likelihood function of Inverse Rayleigh distribution under SSPALT based on APHC for $jth$ ordered lifetime data is given as

$$L^*(p, \beta) \propto \prod_{i=1}^{m_u} f_1(x_{i:m:n})\left[1 - F_1(x_{i:m:n})\right]^{R_i} \left[1 - F_1(\tau)\right]^{R_\tau} \prod_{i=m_u+1}^{j} f_1(x_{i:m:n})\left[1 - F_1(x_{i:m:n})\right]^{R_i} \left[1 - F_1(t_0^*)\right]^{R_J^{**}} \quad (19)$$

Where, $R_j^{**} = (n - \sum_{i=1}^{m} R_i - j)$

$j = m_u + m_a$ , $m_u$ is the number of failed units at use conditions and $m_a$ is the number of failed units at accelerated conditions.

The likelihood function of Inverse Rayleigh distribution under SSPALT based on APHC for $jth$ ordered lifetime data is given as

$$\ln L^* = \sum_{i=1}^{m_u} \ln \frac{2p^2}{x_i^3} - \sum_{i=1}^{m_u} \ln\left(\frac{p}{x_i}\right)^2 + \sum_{i=1}^{m_u} R_i \ln\left[1 - e^{(p/x_i)^2}\right] + R_\tau \sum_{i=1}^{m_u} \ln\left[1 - e^{(p/\tau)^2}\right] +$$
$$+ \sum_{i=m_u+1}^{j} \ln \frac{2\beta p^2}{(\tau + \beta(x_i - \tau))^3} - \sum_{i=m_u+1}^{j} (p/\tau + \beta(x_i - \tau))^2 + \sum_{i=m_u+1}^{j} R_i \ln\left[1 - e^{-(p/\tau+\beta(x_i-\tau))^2}\right] + \sum_{i=m_u+1}^{m} R_J^{**} \ln\left[1 - e^{-\left(p/\tau+\beta(t_0^*-\tau)\right)^2}\right]$$

$$\ln L^* = \sum_{i=1}^{m_u} \ln \frac{2p^2}{x_i^3} - \sum_{i=1}^{m_u} \ln\left(\frac{p}{x_i}\right)^2 + \sum_{i=1}^{m_u} R_i \ln\left[1 - e^{(p/x_i)^2}\right] + R_\tau m_u \ln\left[1 - e^{(p/\tau)^2}\right] +$$
$$+ \sum_{i=m_u+1}^{j} \ln \frac{2\beta p^2}{\psi_i^3} - \sum_{i=m_u+1}^{j} (p/\psi_i)^2 + \sum_{i=m_u+1}^{m} R_i \ln\left[1 - e^{-(p/\psi_i)^2}\right] + m_a R_J^{**} \ln\left[1 - e^{-\left(p/\psi_{t_0^*}\right)^2}\right] \quad (20)$$

Where, $\psi_{t_0^*} = \tau + \beta(t_0^* - \tau)$ , $\ln L^* = \ln L^*(p, \beta)$

Now differentiate the above log-likelihood Equation20 with respect to $p$ and $\beta$ equating both equations to zero.

$$\frac{\partial \ln L^*}{\partial p} = \sum_{i=1}^{m_u} \frac{8p^3}{x_i^6} - \sum_{i=1}^{m_u} \frac{2p^3}{x_i^4} + \sum_{i=1}^{m_u} R_i e^{-(p/x_i)^2} \frac{(2p/x_i)^2}{1-e^{-(p/x_i)^2}} + R_\tau m_u e^{-(p/\tau)^2} \frac{2p/\tau^2}{1-e^{-(p/\tau)^2}} +$$

$$+ \sum_{i=m_u+1}^{j} \frac{8\beta^3 p^2}{\psi_i^6} - \sum_{i=m_u+1}^{j} \frac{2p}{\psi_i^2} + \sum_{i=m_u+1}^{j} \frac{e^{-(p/\psi_i)^2}\, 2p/\psi_i^2}{\left[1-e^{-(p/\psi_i)^2}\right]} R_i + m_a R_J^{**} \frac{e^{-(p/\psi_{t_0})^2}\, 2p/\psi_{t_0^*}^2}{\left[1-e^{-(p/\psi_{t_0})^2}\right]} = 0 \qquad (21)$$

$$\frac{\partial \ln L^*}{\partial \beta} = \sum_{i=m_u+1}^{j} 2p^2 \left[\frac{1}{\beta} - \frac{3(x_i-\tau)}{\tau+\beta(x_i-\tau)}\right] + \sum_{i=m_u+1}^{j} \left[\begin{array}{l} 2p^2(\tau+\beta(x_i-\tau))^{-3} \\ + \sum_{i=m_u+1}^{j} R_i p^2 \dfrac{(x_i-\tau)e^{-(p/\tau+\beta(x_i-\tau))^2}(\tau+\beta(x_i-\tau))^{-3}}{1-e^{-e^{-(p/\tau+\beta(x_i-\tau))^2}}} \end{array}\right] \qquad (22)$$

$$+ m_a R_J^{**} \frac{(t_0^*-\tau)e^{-(p/\tau+\beta(t_0^*-\tau))^2}(\tau+\beta(t_0^*-\tau))^{-3}}{1-e^{-e^{-(p/\tau+\beta(t_0^*-\tau))^2}}} = 0$$

From the above Equation21 and 22, it looks impossible to get a closed-form solution to these equations. So, Newton –Raphson technique is applied to get the ML estimates of $p$ and $\beta$ by using these equations.

## IV. INTERVAL ESTIMATION

On the basis of progressive hybrid censoring (PHC) and adaptive progressive hybrid censoring (APHC), interval estimation for the model parameters and acceleration factor are obtained in this section.

On the basis of data obtained from PHC and APHC schemes, the ML estimates of shape parameter $p$ and acceleration factor $\beta$ are obtained. The asymptotic distribution of ML estimates of $p$ and $\beta$ is given as

$$((\hat{p}-p),(\hat{\beta}-\beta)) \rightarrow N(0, I^{-1}(p,\beta)) \qquad (23)$$

The above technique is suggested by Miller [29]. In the above Equation23, $I^{-1}(p,\beta)$ denotes the variance-covariance matrix of unknown parameters of the model. The elements of $I^{-1}, I_{ij}^{-1}(p,\beta), i=1,2$; approximated by $I^{-1}(p,\beta)$ under PHC and APHC schemes are presented and given in the following sections.

### Fisher Information matrix based on PHC

The Fisher Information matrix under PHC scheme is given as

$$F = \begin{bmatrix} -\dfrac{\partial^2 \ln L}{\partial p^2} & -\dfrac{\partial^2 \ln L}{\partial p \partial \beta} \\ -\dfrac{\partial^2 \ln L}{\partial \beta \partial p} & -\dfrac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}$$

The elements of the Fisher Information matrix under PHC scheme are given as

$$-\frac{\partial^2 \ln L}{\partial p^2} = -\sum_{i=1}^{m_u} \frac{24p^2}{x_i^6} + \sum_{i=1}^{m_u} \frac{6p^2}{x_i^4} - \sum_{i=1}^{m_u} R_i e^{-(p/x_i)^2} \frac{e^{-(p/x_i)^2} x_i^{-2}}{1-e^{-(p/x_i)^2}} \left[\frac{-2p}{x_i^2} + \frac{2px_i^{-2}e^{-(p/x_i)^2}}{1-e^{-(p/x_i)^2}}\right]$$

$$-\frac{2R_\tau m_u}{\tau^2} \frac{e^{-(p/\tau)^2}}{1-e^{-(p/\tau)^2}} \left[\frac{-2p}{\tau} + p^{-1} + \frac{2pe^{-(p/\tau)^2}}{\tau^2(1-e^{-(p/\tau)^2})}\right] - \frac{2R_\tau m_u}{\tau^2} \frac{e^{-(p/\tau)^2}}{1-e^{-(p/\tau)^2}} \left[\frac{-2p}{\tau} + p^{-1} + \frac{2pe^{-(p/\tau)^2}}{\tau^2(1-e^{-(p/\tau)^2})}\right]$$

$$-\sum_{i=m_u+1}^{m} \frac{16\beta^3 p}{\psi_i^6} + \sum_{i=m_u+1}^{m} \frac{2}{\psi_i^2} - \sum_{i=m_u+1}^{m} \frac{e^{-(p/\psi_i)^2}\, 2p/\psi_i^2}{\left[1-e^{-(p/\psi_i)^2}\right]} R_i \left[\frac{-2p}{\psi_i^2} + p^{-1} + \frac{2pe^{-(p/\psi_i)^2}}{\psi_i^2(1-e^{-(p/\psi_i)^2})}\right]$$

$$-2m_a R_J^* \frac{e^{-(p/\psi_t)^2}\psi_t^{-2}}{\left[1-e^{-(p/\psi_i)^2}\right]} \left[\frac{-2p}{\psi_t^2} + \frac{2pe^{-(p/\psi_t)^2}}{\psi_t^2(1-e^{-(p/\psi_t)^2})}\right]$$

$$-\frac{\partial^2 \ln L}{\partial p \partial \beta} = -\sum_{i=m_u+1}^{m} 8p^2\beta^3(\tau+\beta(x_i-\tau))^{-6}\left[\frac{3}{\beta} - \frac{3(x_i-\tau)}{(\tau+\beta(x_i-\tau))}\right] + \sum_{i=m_u+1}^{m} 4p(\tau+\beta(x_i-\tau))^{-3}(x_i-\tau)$$

$$-\sum_{i=m_u+1}^{m} 2R_i p \frac{e^{-(p/\tau+\beta(x_i-\tau))^2}(\tau+\beta(x_i-\tau))^{-2}}{1-e^{-e^{-(p/\tau+\beta(x_i-\tau))^2}}} \left[\begin{array}{l} p^2(\tau+\beta(x_i-\tau))^{-3}(x_i-\tau) - 2(x_i-\tau)(\tau+\beta(x_i-\tau))^{-1} \\ -\dfrac{4pe^{-(p/\tau+\beta(x_i-\tau))^2}(\tau+\beta(x_i-\tau))^{-3}(x_i-\tau)}{1-e^{-(p/\tau+\beta(x_i-\tau))^2}} \end{array}\right]$$

$$-2m_a R_J^* p \frac{e^{-(p/\tau+\beta(t_0-\tau))^2}(\tau+\beta(t_0-\tau))^{-2}}{1-e^{-e^{-(p/\tau+\beta(t_0-\tau))^2}}} \left[\begin{array}{l} p^2(\tau+\beta(t_0-\tau))^{-3}(t_0-\tau) - 2(t_0-\tau)(\tau+\beta(t_0-\tau))^{-1} \\ -\dfrac{4pe^{-(p/\tau+\beta(t_0-\tau))^2}(\tau+\beta(t_0-\tau))^{-3}(t_0-\tau)}{1-e^{-(p/\tau+\beta(t_0-\tau))^2}} \end{array}\right]$$

$$-\frac{\partial^2 \ln L}{\partial p \partial \beta} = -\frac{\partial^2 \ln L}{\partial \beta \partial p}$$

$$-\frac{\partial^2 \ln L}{\partial p \partial \beta} = -\sum_{i=m_u+1}^{m} 8p^2\beta^3(\tau+\beta(x_i-\tau))^{-6}\left[\frac{3}{\beta}-\frac{3(x_i-\tau)}{(\tau+\beta(x_i-\tau))}\right]+\sum_{i=m_u+1}^{m}4p(\tau+\beta(x_i-\tau))^{-3}(x_i-\tau)$$

$$-\sum_{i=m_u+1}^{m}2R_i p\frac{e^{-(p/\tau+\beta(x_i-\tau))^2}(\tau+\beta(x_i-\tau))^{-2}}{1-e^{-e^{-(p/\tau+\beta(x_i-\tau))^2}}}\left[\begin{array}{l} p^2(\tau+\beta(x_i-\tau))^{-3}(x_i-\tau)-2(x_i-\tau)(\tau+\beta(x_i-\tau))^{-1} \\ -\dfrac{4pe^{-(p/\tau+\beta(x_i-\tau))^2}(\tau+\beta(x_i-\tau))^{-3}(x_i-\tau)}{1-e^{-(p/\tau+\beta(x_i-\tau))^2}}\end{array}\right]$$

$$-2m_aR_J^* p\frac{e^{-(p/\tau+\beta(t_0-\tau))^2}(\tau+\beta(t_0-\tau))^{-2}}{1-e^{-e^{-(p/\tau+\beta(t_0-\tau))^2}}}\left[\begin{array}{l} p^2(\tau+\beta(t_0-\tau))^{-3}(t_0-\tau)-2(t_0-\tau)(\tau+\beta(t_0-\tau))^{-1} \\ -\dfrac{4pe^{-(p/\tau+\beta(t_0-\tau))^2}(\tau+\beta(t_0-\tau))^{-3}(t_0-\tau)}{1-e^{-(p/\tau+\beta(t_0-\tau))^2}}\end{array}\right]$$

The approximate $100(1-\alpha)\%$ two-sided confidence limits for shape parameter $p$ and acceleration factor $\beta$, are given as

$$\left\{\begin{array}{l}\hat{p}+Z_{\alpha/2}\sqrt{I_{11}^{-1}(\hat{p},\hat{\beta})}\ and\ \hat{p}-Z_{\alpha/2}\sqrt{I_{11}^{-1}(\hat{p},\hat{\beta})} \\ \hat{p}+Z_{\alpha/2}\sqrt{I_{22}^{-1}(\hat{p},\hat{\beta})}\ and\ \hat{p}-Z_{\alpha/2}\sqrt{I_{22}^{-1}(\hat{p},\hat{\beta})}\end{array}\right\} \quad (24)$$

Where, $Z_{\alpha/2}$ is the upper $(\alpha/2)th$ percentile of standard normal variate.

**Fisher Information matrix based on APHC**

The Fisher Information matrix under APHC is given as

$$F^* = \begin{bmatrix} -\dfrac{\partial^2 \ln L^*}{\partial p^2} & -\dfrac{\partial^2 \ln L^*}{\partial p \partial \beta} \\ -\dfrac{\partial^2 \ln L^*}{\partial \beta \partial p} & -\dfrac{\partial^2 \ln L^*}{\partial \beta^2}\end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22}\end{bmatrix}$$

The elements of the Fisher Information matrix under APHC scheme are given as

$$-\frac{\partial^2 \ln L^*}{\partial p^2} = -\sum_{i=1}^{m_u}\frac{24p^2}{x_i^6}+\sum_{i=1}^{m_u}\frac{6p^2}{x_i^4}-\sum_{i=1}^{m_u}R_i e^{-(p/x_i)^2}\frac{e^{-(p/x_i)^2}x_i^{-2}}{1-e^{-(p/x_i)^2}}\left[\frac{-2p}{x_i^2}+\frac{2px_i^{-2}e^{-(p/x_i)^2}}{1-e^{-(p/x_i)^2}}\right]$$

$$-\frac{2R_\tau m_u}{\tau^2}\frac{e^{-(p/\tau)^2}}{1-e^{-(p/\tau)^2}}\left[\frac{-2p}{\tau}+p^{-1}+\frac{2pe^{-(p/\tau)^2}}{\tau^2(1-e^{-(p/\tau)^2})}\right]$$

$$-\sum_{i=m_u+1}^{j}\frac{16\beta^3 p}{\psi_i^6}+\sum_{i=m_u+1}^{m}\frac{2}{\psi_i^2}-\sum_{i=m_u+1}^{j}\frac{e^{-(p/\psi_i)^2}\,2p/\psi_i^2}{\left[1-e^{-(p/\psi_i)^2}\right]}R_i\left[\frac{-2p}{\psi_i^2}+p^{-1}+\frac{2pe^{-(p/\psi_i)^2}}{\psi_i^2(1-e^{-(p/\psi_i)^2})}\right]$$

$$-2m_aR_J^{**}\frac{e^{-(p/\psi_t)^2}\psi_{t_0^*}^{-2}}{\left[1-e^{-\left(p/\psi_{t_0^*}\right)^2}\right]}\left[\frac{-2p}{\psi_{t_0^*}^2}+\frac{2pe^{-(p/\psi_{t_0^*})^2}}{\psi_{t_0^*}^2(1-e^{\left(p/\psi_{t_0^*}\right)^2})}\right]$$

$$-\frac{\partial^2 \ln L^*}{\partial \beta^2} = -\sum_{i=m_u+1}^{j}2p^2\left[-\frac{1}{\beta^2}-\frac{3(x_i-\tau)^2}{(\tau+\beta(x_i-\tau))^2}\right]+\sum_{i=m_u+1}^{j}6p^2(\tau+\beta(x_i-\tau))^{-4}(x_i-\tau)$$

$$-\sum_{i=m_u+1}^{j}R_i p^2\frac{(x_i-\tau)e^{-(p/\tau+\beta(x_i-\tau))^2}(\tau+\beta(x_i-\tau))^{-3}}{1-e^{-e^{-(p/\tau+\beta(x_i-\tau))^2}}}\left[\begin{array}{l}2p^2(\tau+\beta(x_i-\tau))^{-3}-3(x_i-\tau)(\tau+\beta(x_i-\tau))^{-1} \\ -\dfrac{2p^2e^{-(p/\tau+\beta(x_i-\tau))^2}(\tau+\beta(x_i-\tau))(x_i-\tau)}{1-e^{-(p/\tau+\beta(x_i-\tau))^2}}\end{array}\right]$$

$$-m_aR_J^{**}p^2\frac{(t_0^*-\tau)e^{-(p/\tau+\beta(t_0^*-\tau))^2}(\tau+\beta(t_0^*-\tau))^{-3}}{1-e^{-e^{(p/\tau+\beta(t_0^*-\tau))}}}\left[\begin{array}{l}2p^2(\tau+\beta(t_0^*-\tau))^{-3}-3(t_0^*-\tau)(\tau+\beta(t_0^8-\tau))^{-1} \\ -\dfrac{2p^2e^{-(p/\tau+\beta(t_0^*-\tau))^2}(\tau+\beta(t_0^*-\tau))(t_0^*-\tau)}{1-e^{-(p/\tau+\beta(t_0^*-\tau))^2}}\end{array}\right]$$

$$-\frac{\partial^2 \ln L^*}{\partial p \partial \beta} = -\frac{\partial^2 \ln L^*}{\partial \beta \partial p}$$

$$-\frac{\partial^2 \ln L}{\partial p \partial \beta} = -\sum_{i=m_u+1}^{j} 8p^2\beta^3(\tau+\beta(x_i-\tau))^{-6}\left[\frac{3}{\beta}-\frac{3(x_i-\tau)}{(\tau+\beta(x_i-\tau))}\right] + \sum_{i=m_u+1}^{j} 4p(\tau+\beta(x_i-\tau))^{-3}(x_i-\tau)$$

$$-\sum_{i=m_u+1}^{j} 2R_i p \frac{e^{-(p/\tau+\beta(x_i-\tau))^2}(\tau+\beta(x_i-\tau))^{-2}}{1-e^{-e^{-(p/\tau+\beta(x_i-\tau))^2}}}\left[\begin{array}{l} p^2(\tau+\beta(x_i-\tau))^{-3}(x_i-\tau)-2(x_i-\tau)(\tau+\beta(x_i-\tau))^{-1} \\ -\dfrac{4pe^{-(p/\tau+\beta(x_i-\tau))^2}(\tau+\beta(x_i-\tau))^{-3}(x_i-\tau)}{1-e^{-(p/\tau+\beta(x_i-\tau))^2}} \end{array}\right]$$

$$-2m_a R_J^{**} p \frac{e^{-(p/\tau+\beta(t_0^*-\tau))^2}(\tau+\beta(t_0^*-\tau))^{-2}}{1-e^{-e^{-(p/\tau+\beta(t_0^*-\tau))^2}}}\left[\begin{array}{l} p^2(\tau+\beta(t_0^*-\tau))^{-3}(t_0^*-\tau)-2(t_0^*-\tau)(\tau+\beta(t_0^*-\tau))^{-1} \\ -\dfrac{4pe^{-(p/\tau+\beta(t_0^*-\tau))^2}(\tau+\beta(t_0^*-\tau))^{-3}(t_0^*-\tau)}{1-e^{-(p/\tau+\beta(t_0^*-\tau))^2}} \end{array}\right]$$

The approximate $100(1-\alpha)\%$ two-sided confidence limits for shape parameter $p$ and acceleration factor $\beta$, are given as

$$\left\{\begin{array}{l} \hat{p}+Z_{\alpha/2}\sqrt{I_{11}^{-1}(\hat{p},\hat{\beta})} \ and \ \hat{p}-Z_{\alpha/2}\sqrt{I_{11}^{-1}(\hat{p},\hat{\beta})} \\ \hat{p}+Z_{\alpha/2}\sqrt{I_{22}^{-1}(\hat{p},\hat{\beta})} \ and \ \hat{p}-Z_{\alpha/2}\sqrt{I_{22}^{-1}(\hat{p},\hat{\beta})} \end{array}\right\} \quad (25)$$

## V. Simulation Studies

In this section, the Monte-Carlo simulation technique is used to compare the performance of different censoring schemes for different values of the parameters. Also, the performance of the MLEs is compared in terms of their mean square error and biases on the basis of the parameter values of two different censoring schemes (PHC and APHC). Here we take three progressive censoring schemes. On the basis of 1000 simulations and for each scheme, the biases and (Mean Square Errors) MSEs are estimated. The three progressive censoring schemes are

Scheme I: $R_1 = R_2 = ... = R_{m-1}$ and $R_m = n-m$

Scheme II: $R_1 = n-m$ and $R_2 = R_3 = ... = R_m = 0$

Scheme III: $R_1 = R_2 = ... = R_{m-1}$ and $R_m = n-2m+1$

The simulation steps for the above three censoring schemes are given as

(i) Specify the values of parameters $n, m, \tau$ and $t_0^*$.

(ii) Also, specify the values of model parameters $p$ and $\beta$.

(iii) By inverse CDF method, generate random samples from Inverse Rayleigh distribution in both cases, i.e., normal and accelerated conditions.

(iv) Generate the progressive hybrid censored sample using the model given in Equation6 for given values of parameters $n, m.\tau, t_0^*, p$ and $\beta$. The sample data sets are given as

Set 1 (PHC): $x_{1:m:n} < ... < x_{m_u} \le \tau < x_{m_u+1} < ... < x_{m:m:n}$

Set 2 (AHPC): $x_{1:m:n} < ... < x_{m_u} \le \tau < x_{m_u+1:m:n} < ... < x_{m:m:n} < x_{m+1:m:n} < ... < x_{j:m:n} \le t_0^*$

(v) To compute the MLEs of the distribution parameters, we use progressive hybrid censored samples.

(vi) To compute the MLEs of the parameters, an iterative technique (Newton-Raphson) is used to solve the non-linear equations.

(vii) Steps (iii)-(vi) replicate 1000 times.

(viii) Compute the average values of MSEs and biases.

Table 5.1: Mean values of MSEs and biases under PHC and APHC, when $p, \beta, \tau$ and $t_0^*$ are fixed at 0.6, 1.4, 2.8 and 8

| $(n.m)$ | Schemes | Bias of $p$ | | MSE of $p$ | | Bias of $\beta$ | | MSE of $\beta$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | PHC | APHC | PHC | APHC | PHC | APHC | PHC | APHC |
| (30,10) | 1 | 0.712 | 0.527 | 0.843 | 0.616 | 0.826 | 0.623 | 0.835 | 0.732 |
| | 2 | 0.765 | 0.645 | 0.886 | 0.746 | 0.885 | 0.765 | 0.891 | 0.764 |
| | 3 | 0.743 | 0.592 | 0.867 | 0.587 | 0.854 | 0.686 | 0.868 | 0.749 |
| (40,10) | 1 | 0.612 | 0.464 | 0.764 | 0.565 | 0.712 | 0.567 | 0.712 | 0.633 |
| | 2 | 0.652 | 0.565 | 0.779 | 0.665 | 0.765 | 0.643 | 0.765 | 0.654 |
| | 3 | 0.631 | 0.485 | 0.785 | 0.576 | 0.732 | 0.597 | 0.732 | 0.649 |
| (60,10) | 1 | 0.512 | 0.375 | 0.512 | 0.443 | 0.586 | 0.451 | 0.632 | 0.521 |
| | 2 | 0.576 | 0.445 | 0.875 | 0.546 | 0.687 | 0.576 | 0.797 | 0.576 |
| | 3 | 0.543 | 0.397 | 0.586 | 0.487 | 0.654 | 0.483 | 0.654 | 0.565 |
| (30,16) | 1 | 0.476 | 0.314 | 0.398 | 0.365 | 0.476 | 0.334 | 0.542 | 0.412 |
| | 2 | 0.498 | 0.365 | 0.576 | 0.432 | 0.554 | 0.453 | 0.599 | 0.484 |
| | 3 | 0.488 | 0.325 | 0.510 | 0.397 | 0.522 | 0.376 | 0.576 | 0.474 |
| (40,16) | 1 | 0.324 | 0.235 | 0.317 | 0.276 | 0.391 | 0.265 | 0.451 | 0.325 |
| | 2 | 0.387 | 0.276 | 0.486 | 0.354 | 0.443 | 0.335 | 0.512 | 0.411 |
| | 3 | 0.367 | 0.254 | 0.421 | 0.300 | 0.408 | 0.287 | 0.486 | 0.387 |
| (60,16) | 1 | 0.298 | 0.164 | 0.265 | 0.218 | 0.303 | 0.175 | 0.364 | 0.243 |
| | 2 | 0.322 | 0.198 | 0.554 | 0.264 | 0.365 | 0.276 | 0.532 | 0.323 |
| | 3 | 0.302 | 0.127 | 0.338 | 0.231 | 0.323 | 0.197 | 0.385 | 0.287 |

Table 5.2: Mean values of MSEs and biases under PHC and APHC, when $p, \beta, \tau$ and $t_0^*$ are fixed at 0.6, 1.4, 2.8 and 14

| $(n.m)$ | Schemes | Bias of $p$ | | MSE of $p$ | | Bias of $\beta$ | | MSE of $\beta$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | PHC | APHC | PHC | APHC | PHC | APHC | PHC | APHC |
| (30,10) | 1 | 0.532 | 0.445 | 0.654 | 0.564 | 0.587 | 0.523 | 0.698 | 0.632 |
| | 2 | 0.612 | 0.534 | 0.708 | 0.643 | 0.653 | 0.588 | 0.765 | 0.687 |
| | 3 | 0.564 | 0.486 | 0.674 | 0.597 | 0.623 | 0.564 | 0.745 | 0.665 |
| (40,10) | 1 | 0.454 | 0.386 | 0.564 | 0.475 | 0.475 | 0.422 | 0.611 | 0.543 |
| | 2 | 0.542 | 0.453 | 0.613 | 0.564 | 0.576 | 0.511 | 0.654 | 0.587 |
| | 3 | 0.487 | 0.412 | 0.598 | 0.523 | 0.532 | 0.486 | 0.633 | 0.554 |
| (60,10) | 1 | 0.364 | 0.312 | 0.485 | 0.384 | 0.365 | 0.312 | 0.543 | 0.454 |
| | 2 | 0.443 | 0.387 | 0.714 | 0.475 | 0.498 | 0.443 | 0.787 | 0.499 |
| | 3 | 0.404 | 0.345 | 0.497 | 0.431 | 0.454 | 0.398 | 0.565 | 0.487 |
| (30,16) | 1 | 0.286 | 0.243 | 0.365 | 0.297 | 0.313 | 0.274 | 0.465 | 0.334 |
| | 2 | 0.376 | 0.298 | 0.421 | 0.386 | 0.408 | 0.354 | 0.498 | 0.409 |
| | 3 | 0.298 | 0.265 | 0.385 | 0.342 | 0.376 | 0.321 | 0.469 | 0.392 |
| (40,16) | 1 | 0.187 | 0.165 | 0.254 | 0.174 | 0.224 | 0.175 | 0.376 | 0.211 |
| | 2 | 0.297 | 0.221 | 0.304 | 0.296 | 0.321 | 0.265 | 0.510 | 0.343 |
| | 3 | 0.221 | 0.186 | 0.275 | 0.264 | 0.276 | 0.232 | 0.398 | 0.288 |
| (60,16) | 1 | 0.112 | 0.110 | 0.175 | 0.097 | 0.154 | 0.0.07 | 0.298 | 0.140 |
| | 2 | 0.204 | 0.143 | 0.321 | 0.176 | 0.213 | 0.176 | 0.354 | 0.232 |
| | 3 | 0.153 | 0.121 | 0.197 | 0.143 | 0.197 | 0.142 | 0.321 | 0.176 |

Table 5.3: Mean values of MSEs and biases under PHC and APHC, when $p, \beta, \tau$ and $t_0^*$ are fixed at 1.6, 1.4, 2.8 and 8

| $(n.m)$ | Schemes | Bias of $p$ | | MSE of $p$ | | Bias of $\beta$ | | MSE of $\beta$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | PHC | APHC | PHC | APHC | PHC | APHC | PHC | APHC |
| (30,10) | 1 | 0.616 | 0.543 | 0.677 | 0.576 | 0.645 | 0.567 | 0.687 | 0.655 |
| | 2 | 0.698 | 0.588 | 0.722 | 0.645 | 0.745 | 0.688 | 0.812 | 0.740 |
| | 3 | 0.655 | 0.566 | 0.687 | 0.618 | 0.687 | 0.614 | 0.765 | 0.672 |
| (40,10) | 1 | 0.523 | 0.476 | 0.623 | 0.497 | 0.586 | 0.517 | 0.645 | 0.554 |
| | 2 | 0.633 | 0.523 | 0.654 | 0.587 | 0.687 | 0.577 | 0.753 | 0.674 |
| | 3 | 0.587 | 0.498 | 0.630 | 0.532 | 0.593 | 0.543 | 0.707 | 0.587 |
| (60,10) | 1 | 0.408 | 0.365 | 0.509 | 0.406 | 0.465 | 0.365 | 0.512 | 0.487 |
| | 2 | 0.575 | 0.465 | 0.587 | 0.443 | 0.608 | 0.495 | 0.653 | 0.589 |
| | 3 | 0.465 | 0.387 | 0.541 | 0.411 | 0.523 | 0.463 | 0.628 | 0.520 |
| (30,16) | 1 | 0.345 | 0.299 | 0.443 | 0.365 | 0.398 | 0.274 | 0.433 | 0.404 |
| | 2 | 0.443 | 0.344 | 0.456 | 0.380 | 0.521 | 0.385 | 0.565 | 0.518 |
| | 3 | 0.387 | 0.321 | 0.466 | 0.326 | 0.440 | 0.332 | 0.517 | 0.474 |
| (40,16) | 1 | 0.265 | 0.177 | 0.365 | 0.276 | 0.316 | 0.143 | 0.319 | 0.365 |
| | 2 | 0.321 | 0.276 | 0.388 | 0.311 | 0.476 | 0.298 | 0.487 | 0.470 |
| | 3 | 0.312 | 0.254 | 0.371 | 0.253 | 0.365 | 0.248 | 0.384 | 0.418 |
| (60,16) | 1 | 0.188 | 0.065 | 0.243 | 0.187 | 0.202 | 0.043 | 0.245 | 0.315 |
| | 2 | 0.232 | 0.199 | 0.297 | 0.254 | 0.387 | 0.234 | 0.387 | 0.407 |
| | 3 | 0.204 | 0.123 | 0.276 | 0.207 | 0.283 | 0.177 | 0.288 | 0.358 |

Table 5.4: Mean values of MSEs and biases under PHC and APHC, when $p, \beta, \tau$ and $t_0^*$ are fixed at 1.6, 1.4, 2.8 and 14

| $(n.m)$ | Schemes | Bias of $p$ | | MSE of $p$ | | Bias of $\beta$ | | MSE of $\beta$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | PHC | APHC | PHC | APHC | PHC | APHC | PHC | APHC |
| (30,10) | 1 | 0.434 | 0.376 | 0.482 | 0.391 | 0.504 | 0.482 | 0.570 | 0.485 |
| | 2 | 0.487 | 0.408 | 0.539 | 0.425 | 0.548 | 0.511 | 0.609 | 0.540 |
| | 3 | 0.422 | 0.392 | 0.520 | 0.400 | 0.619 | 0.493 | 0.594 | 0.525 |
| (40,10) | 1 | 0.387 | 0.297 | 0.418 | 0.324 | 0.456 | 0.392 | 0.521 | 0.416 |
| | 2 | 0.429 | 0.344 | 0.483 | 0.379 | 0.490 | 0.466 | 0.687 | 0.488 |
| | 3 | 0.410 | 0.329 | 0.460 | 0.368 | 0.476 | 0.421 | 0.544 | 0.447 |
| (60,10) | 1 | 0.318 | 0.234 | 0.346 | 0.254 | 0.367 | 0.329 | 0.453 | 0.345 |
| | 2 | 0.354 | 0.265 | 0.506 | 0.317 | 0.438 | 0.372 | 0.506 | 0.407 |
| | 3 | 0.343 | 0.254 | 0.437 | 0.298 | 0.417 | 0.352 | 0.479 | 0.398 |
| (30,16) | 1 | 0.232 | 0.165 | 0.288 | 0.193 | 0.298 | 0.250 | 0.377 | 0.276 |
| | 2 | 0.268 | 0.199 | 0.348 | 0.245 | 0.333 | 0.315 | 0.458 | 0.342 |
| | 3 | 0.258 | 0.173 | 0.343 | 0.229 | 0.328 | 0.265 | 0.393 | 0.280 |
| (40,16) | 1 | 0.142 | 0.087 | 0.233 | 0.125 | 0.219 | 0.165 | 0.287 | 0.219 |
| | 2 | 0.190 | 0.121 | 0.276 | 0.180 | 0.288 | 0.237 | 0.465 | 0.304 |
| | 3 | 0.179 | 0.090 | 0.268 | 0.168 | 0.260 | 0.194 | 0.307 | 0.274 |
| (60,16) | 1 | 0.100 | 0.056 | 0.156 | 0.067 | 0.157 | 0.005 | 0.188 | 0.154 |
| | 2 | 0.141 | 0.088 | 0.318 | 0.104 | 0.238 | 0.160 | 0.279 | 0.237 |
| | 3 | 0.122 | 0.065 | 0.200 | 0.088 | 0.177 | 0.100 | 0.233 | 0.180 |

Table 5.5: Mean values of MSEs and biases under PHC and APHC, when $p, \beta, \tau$ and $t_0^*$ are fixed at 1.6, 1.4, 3.2 and 8

| $(n.m)$ | Schemes | Bias of $p$ | | MSE of $p$ | | Bias of $\beta$ | | MSE of $\beta$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | PHC | APHC | PHC | APHC | PHC | APHC | PHC | APHC |
| (30,10) | 1 | 0.398 | 0.314 | 0.449 | 0.361 | 0.503 | 0.421 | 0.549 | 0.466 |
| | 2 | 0.457 | 0.360 | 0.531 | 0.475 | 0.580 | 0.499 | 0.608 | 0.530 |
| | 3 | 0.427 | 0.344 | 0.494 | 0.439 | 0.566 | 0.469 | 0.587 | 0.499 |
| (40,10) | 1 | 0.325 | 0.246 | 0.360 | 0.280 | 0.411 | 0.340 | 0.472 | 0.386 |
| | 2 | 0.387 | 0.289 | 0.467 | 0.388 | 0.492 | 0.404 | 0.544 | 0.465 |
| | 3 | 0.365 | 0.286 | 0.417 | 0.342 | 0.455 | 0.387 | 0.489 | 0.408 |
| (60,10) | 1 | 0.240 | 0.163 | 0.287 | 0.204 | 0.327 | 0.275 | 0.397 | 0.300 |
| | 2 | 0.328 | 0.208 | 0.380 | 0.334 | 0.419 | 0.331 | 0.461 | 0.376 |
| | 3 | 0.297 | 0.198 | 0.341 | 0.259 | 0.368 | 0.319 | 0.421 | 0.322 |
| (30,16) | 1 | 0.174 | 0.087 | 0.218 | 0.138 | 0.246 | 0.205 | 0.302 | 0.211 |
| | 2 | 0.259 | 0.125 | 0.308 | 0.274 | 0.341 | 0.258 | 0.379 | 0.287 |
| | 3 | 0.210 | 0.117 | 0.264 | 0.183 | 0.290 | 0.222 | 0.334 | 0.229 |
| (40,16) | 1 | 0.104 | 0.037 | 0.137 | 0.073 | 0.154 | 0.128 | 0.216 | 0.148 |
| | 2 | 0.178 | 0.085 | 0.229 | 0.180 | 0.260 | 0.169 | 0.295 | 0.208 |
| | 3 | 0.132 | 0.067 | 0.181 | 0.090 | 0.218 | 0.144 | 0.250 | 0.152 |
| (60,16) | 1 | 0.005 | 0.009 | 0.072 | 0.004 | 0.084 | 0.007 | 0.140 | 0.073 |
| | 2 | 0.118 | 0.016 | 0.151 | 0.089 | 0.174 | 0.117 | 0.218 | 0.121 |
| | 3 | 0.009 | 0.010 | 0.103 | 0.009 | 0.141 | 0.101 | 0.166 | 0.109 |

Table 5.6: Mean values of MSEs and biases under PHC and APHC, when $p, \beta, \tau$ and $t_0^*$ are fixed at 1.6, 1.4, 3.2 and 14

| $(n.m)$ | Schemes | Bias of $p$ | | MSE of $p$ | | Bias of $\beta$ | | MSE of $\beta$ | |
|---|---|---|---|---|---|---|---|---|---|
| | | PHC | APHC | PHC | APHC | PHC | APHC | PHC | APHC |
| (30,10) | 1 | 0.481 | 0.404 | 0.599 | 0.514 | 0.603 | 0.516 | 0.680 | 0.567 |
| | 2 | 0.547 | 0.497 | 0.632 | 0.585 | 0.678 | 0.595 | 0.730 | 0.600 |
| | 3 | 0.501 | 0.444 | 0.600 | 0.539 | 0.566 | 0.569 | 0.632 | 0.590 |
| (40,10) | 1 | 0.415 | 0.361 | 0.510 | 0.473 | 0.531 | 0.440 | 0.576 | 0.487 |
| | 2 | 0.471 | 0.382 | 0.567 | 0.500 | 0.582 | 0.498 | 0.745 | 0.518 |
| | 3 | 0.453 | 0.386 | 0.527 | 0.449 | 0.505 | 0.443 | 0.619 | 0.500 |
| (60,10) | 1 | 0.343 | 0.273 | 0.475 | 0.354 | 0.417 | 0.375 | 0.497 | 0.401 |
| | 2 | 0.387 | 0.328 | 0.504 | 0.434 | 0.499 | 0.412 | 0.561 | 0.461 |
| | 3 | 0.399 | 0.288 | 0.487 | 0.391 | 0.448 | 0.369 | 0.521 | 0.422 |
| (30,16) | 1 | 0.265 | 0.197 | 0.386 | 0.286 | 0.336 | 0.315 | 0.420 | 0.319 |
| | 2 | 0.319 | 0.254 | 0.528 | 0.384 | 0.401 | 0.365 | 0.579 | 0.371 |
| | 3 | 0.291 | 0.209 | 0.410 | 0.332 | 0.389 | 0.322 | 0.434 | 0.329 |
| (40,16) | 1 | 0.214 | 0.137 | 0.297 | 0.193 | 0.254 | 0.228 | 0.361 | 0.248 |
| | 2 | 0.278 | 0.185 | 0.582 | 0.289 | 0.366 | 0.269 | 0.450 | 0.308 |
| | 3 | 0.225 | 0.177 | 0.311 | 0.292 | 0.297 | 0.254 | 0.350 | 0.252 |
| (60,16) | 1 | 0.154 | 0.099 | 0.212 | 0.094 | 0.176 | 0.157 | 0.249 | 0.183 |
| | 2 | 0.208 | 0.126 | 0.351 | 0.199 | 0.274 | 0.227 | 0.518 | 0.201 |
| | 3 | 0.196 | 0.110 | 0.253 | 0.119 | 0.207 | 0.201 | 0.206 | 0.219 |

From Table 5.1-5.6, the MLEs of the distribution parameters based on PHC give larger MSEs compared to those based on APHC. As the effective sample size (m) increases, the MSEs of the MLEs of the two parameters based on APHC and PHC decreases in all cases except for some cases of scheme 2 under PHC. This change in scheme 2 under PHC is occurring due to the weighty censoring at the starting stage of the test. The values of biases under APHC are smaller in comparison to PHC. The following results are made from the above tables

(i) For fixed values of $n, \tau$ and $t_0^*$, the values of MSEs and biases decrease as $m$ increases.

(i) The values of MSEs and biases decrease as $n$ and $m$ increase at the same time.

(iii) The values of MSEs and biases decrease as $t_0^*$ increases.

## VI. **CONCLUSIONS**

This paper is based on Type-I hybrid censoring and adaptive Type-I hybrid censoring and also analyze failure time data using maximum likelihood technique under SSPALT when the lifetime of test units follows Inverse Rayleigh distribution. The numerical values of MLEs of model parameters are obtained using the Newton-Raphson technique, and their performance is analyzed in terms of MLEs and biases. Under APHC, a greater efficiency is obtained in the estimation of model parameters because an ample amount of failures can be observed. In general, the APHC is a good choice to obtain greater efficiency of estimates of parameters for the experimenter, if there is no concerned about time. If time is limited, then PHC is a good choice.

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