NON-EXPANSIVE MAPPINGS IN BANACH LATTICES

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ABSTRACT

In this paper we will prove the existence of a fixed point for a non-expansive mapping operating in a convex subset of Banach lattice E compact for some natural topology τ on E. In particular, if E is a Banach space with a l-unconditional basis we can take for τ the topology of coordinate wise convergence.

Key words: Metric space, complete metric space, expansive mapping, non expansive mapping, Lattice, Banach Lattices.

INTRODUCTION

If B is a subset of a Banach space, a map T : B → B is said to be non-expensive when the inequality

$$|| T(x) – T(y) || \le || x – y ||$$

holds for pair x, y in B.

The main result of this section is the following one. Let E be a Banach space endowed with a l-unconditional Schauder basis, i.e., a Schauder basis $(e_n)_{n\ge 0}$ such that $|| x_0e_0 + \ldots + x_ne_n + \ldots || \le || y_0e_0 + \ldots + y_ne_n + \ldots ||$ if $| x_n | \le | y_n |$ for every n (in fact, this condition will be slightly weakened). Let B be a convex non-void subset of E, compact for the topology of coordinate wise convergence. Then every no-expensive map T : B → B has a fixed point.

This was proved by Lin (1985) in the special case where B is weakly compact convex set. The method of Lin (1985) is a refinement of techniques of Maurey (1981) and Elton *et al.* (1983). It turns out that this method still works with the topology of coordinate wise convergence once a key lemma of Goebel (1975) and Karlovitz (1976) has been generalized. Let us notice that our proof avoids any use of ultra-products.

In fact, in the theorem below, we give a more general result, considering the arbitrary Banach lattices E and proving the above fixed point property in convex subsets compact for some natural topology τ on E. In usual spaces of measurable functions with order continuous norm, τ is the topology of convergence in measure on every set with finite measure.

Related results can be found in Borwein and Sims (1984) for weakly compact convex subsets of Banach lattices and in Lami-Dozo and Turpin (1987) for τ compact star shaped subsets of Orlicz spaces. The techniques of Lami-Dozo and Turpin (1987) using the unicity of "asymptotic centers" exclude such spaces as c_o. In Elton *et al.* (1983) and Borwein and Sims (1984) the norm must be in some sense far from that of l_1. The method of Lin (1985) is generalized here, avoids these restrictions.

A survey on fixed points and no-expansive is also given in Kirk (1983).

Let us recall that Alspach (1981) constructed a weakly compact convex subset of L_1 (0, 1) invariant under some non-expansive map without fixed point.

Theorem:

Let (E , || . ||) be a real Banach space, endowed with a vector lattice structure satisfying

(α) $(x^+ \le y^+ \text{ and } x^- \le y^-) \Rightarrow || x || \le || y ||$. x, y in E,

and, for some real constant k < 2.

(β) $| x | \le | y | \Rightarrow || x || \le k || y ||$, x, y in E.

Let τ be the coarsest linear topology on E for which the map $x \to || \, | x | \wedge u \, ||$ from E to R^+ is continuous at 0 for every $u \in E^+$

Let B be a τ compact nonvoid convex subset of E.

Then every non-expansive map T : B → B has a fixed point.

The above topology τ may also be defined in the following way: it admits as a basis of neighbourhoods of a point x ∈ E the sets

$\{y \in E : || \, | y - x | \wedge u \, || \leq \varepsilon\}$, $u \in E^{+}$. $\varepsilon > 0$.

For example let E be a real Banach space endowed with an unconditional Schauder basis $(e_n)_{n\geq 0}$. Then E be a vector lattice for the "coordinate-wise order" defined by $\sum_n x_n e_n \leq \sum_n y_n e_n$ when $x_n \leq y_n$ for every $n \geq 0$. The topology τ is easily seen to be the topology of coordinate wise convergence. Putting $x = \sum_{n=0}^{\infty} x_n e_n$ the conditions (α) and (β) are respectively equivalent to

(α') $|| \sum_{n=0}^{\infty} \varepsilon_n x_n e_n || \leq || x ||$ $\varepsilon_n = 0, 1, x \in E$.

(β') $|| \sum_{n=0}^{\infty} \varepsilon_n x_n e_n || \leq k \, || x ||$ $\varepsilon_n = \pm 1, x \in E$.

For instance, (α') implies (α) because, if $0 \leq t_n \leq 1$ from every n , $\sum_{n=0}^{\infty} t_n x_n e_n$ lies in the closed convex hull of the points $\sum_{n=0}^{\infty} \varepsilon_n x_n e_n$. $\varepsilon_n = 0, 1$.

So we get the following special case of the theorem.

Corollary 1:

Let E be a real Banach space endowed with an unconditional Schauder basis satisfying the above conditions (α') and (β'), with k < 2. Let B be a non-void convex subset of E, compact for the topology of coordinate wise convergence.

Then every non-expensive map T : B → B has a fixed point.

When B is weakly compact, the above statement is due to Lin [Ln].

Example 1:

Let B be a novoid convex subset of the space $l_1(N)$ of absolutely summable sequences, weak*-compact for the usual duality with $c_0(N)$ (or, equivalently, coordinate wise compact for the canonical basis (e_n) of $l_1(N)$). Then B has a fixed point for a mapping T : B → B if T is non-expensive for the norm

$p_\lambda(x) = || x ||_1 \vee (\lambda \, || x ||_\infty)$

where λ is some real number, $|| x_n ||_1 = \sum_0^{\infty} | x_n |$, $|| x_n ||_\infty = \sup_n | x_n |$.

This result is evident for the weakly compact convex sets since these sets are in fact compact in norm, but we see no obvious way to deduce it from known results for arbitrary weak*-compact convex set if λ ≥ 2.

For instance, let us consider the weak*-compact convex set $B = \{ x \in l_1^{+} : || x ||_1 \leq 1\}$. It contains the unit vectors e_n , n ≥ 0. If λ ≥ 2, then, for every $x \in B$, $p_\lambda(x - e_n)$ tends to the diameter λ of B as n → ∞. This shows that B has no weak*-normal structure (Kirk, 1983), if λ≥ 2. So the methods using normal structures cannot be applied in this case.

On the other hand, Borwein and Sims (1984) generalized the technique of Maurey (1981) to some Banach lattices E with a "Riesz angle" $\alpha(E) = \sup \{|| x || \vee || y || : || x || \vee || y || \leq 1\}$ satisfying α(E)<2. The Riesz angle of (l_1, p_λ) is equal to 2, so the space (l_1, p_λ) does not fall under the scope of Borwein and Sims (1984). Moreover only weakly compact convex sets are considered in Borwein and Sims (1984). However, let us consider the Banach lattice c(N) of convergent sequences, endowed with the supermum norm. It is proved in Borwein and Sims (1984) that every weakly compact convex subset of c(N) has the "non-expensive fixed point property". This is not given by our theorem. Indeed the topology of τ of c(N) is the norm-topology.

Let us observe that the conclusion of Corollary 1 is false for k = 2, as shown by an example of Lin (1980, 1985).

Corollary 1 is concerned with sequence spaces. It can be generalized to spaces of measurable functions in the following way.

Let (Ω, A, μ) be a measure space and let M = M(Ω, A, μ) be the vector lattice of all μ-classes of μ-measurable functions on (Ω, A, μ). An order ideal of M is the vector subspace E of M such that x ∈ E as soon as x ∈ M and | x | ≤ | y | for some element y of E. A norm || . || on E is said to be sequentially order continuous when $\lim_n || x_n || = 0$ for every decreasing sequence (x_n) of E^{+} with $\inf_n x_n = 0$.

Corollary 2:

Let E be an order ideal of M(Ω, A, μ) and a Banach space for a sequentially order continuous norm satisfying conditions (α) and (β) of the theorem, for some constant k < 2. We assume also that an element x of E is null provided x = 0 i.e., on every set A ∈ A with finite measure. Let B be a nonvoid convex subset of E, compact for the topology σ of convergence in measure on every set A ∈ A with finite measure. Then every non-expensive map T : B → B has a fixed point.

This topology σ on E admits as a basis of neighbourhoods of a point $x_0 \in E$ the family of sets

$V(x_o, A, a, \varepsilon) = \{ x \in E : \mu(A \cap (|x - x_0| > a)) < \varepsilon \}$.

Where $a > 0$, $\varepsilon > 0$ and $A \in A$ with $\mu(A) < \infty$. When μ is the counting measure on the power set 2^Γ of a nonvoid set Γ, σ is the topology of point wise convergence.

Proof of Corollary 2:

It suffices to check that σ is finer than τ, i.e., that the mapping $x \to \| |x| \wedge u \|$ from (E, σ) to R^+ is continuous at 0 for every $u \in E^+$, and to apply the theorem. So, let $u \in E^+$, $u \neq 0$ and let $(A_i)_{i\in I}$ be a maximal disjoint family of sets $A_i \in A$ such that $\mu(A_i) < \infty$ and $\| u1_{A_i} \| > 0$, where l_{A_i}, is the characteristic function of A_i. Using the sequential order continuity of the norm, it is easily checked that I is finite or countable. Applying again this property one can find a set $A \in A$ for which $\mu(A) < \infty$ and $\|u - u1_A\| < \varepsilon$, where $\varepsilon > 0$ is given. One can also find real numbers $a > 0$, $b > 0$, $\eta > 0$ satisfying $\|u1_{(u<a)}\| < \varepsilon$, $\|bu\| < \varepsilon$ and $\|u1_B\| < \varepsilon$ for any $B \in A$ with $\mu(B) < \eta$. Then, if $x \in E$ and $\mu(A \cap (|x| > ab)) < \eta$, we get (using condition (α)) $\| |x| \wedge u \| \leq \| u - u1_A \| + \|u1_{(u<a)}\| + \|bu\| + \|u1_{A \cap (|x| > ab)}\| \leq 4\varepsilon$.

Example 2:

Let $L_\Phi = L_\Phi(\Omega, A, \mu)$ be an Orlicz space, Φ being a convex Orlicz function. In corollary 2 we can take for E the closed linear subspace L_Φ^0 of L_Φ generated by the μ-integrable simple functions endowed with any Reisz norm equivalent to the usual Luxemberg norm (in this case the topologies σ and τ are identical).

Let us observe that if Φ verifies the condition Δ_2 (i.e., if $L_\Phi^0 = L_\Phi$) and if $E = L_\Phi$ is endowed with the Luxemberg norm, the set B of corollary 2 need not be convex: it suffices to suppose B star-shaped bounded in norm and compact for the topology σ.

Lemma 1:

(a) The topology τ is a Hausdroff linear topology, coarser that the topology defined by the norm.

(b) Every convex τ compact subset C of E is norm-bounded.

(c) If K is a τ compact subset of E, every sequence (x_n) of points of K contains a τ convergent sub-sequence.

Proof of (a):

Obvious.

Proof of (b):

When C is nonvoid, let E_C be the vector subspace of E generated by C. Clearly, the functional $\| x \|_C = \inf\{t > 0: x \in t(C - C)\}$, $x \in E_C$, ius a norm on E_C, the canonical injection of $(E_C, \| . \|_C)$ into (E, τ) is continuous and $(E_C, \| . \|_C)$ is a Banach space, if $(x_n) \subset C - C$ is a Cauchy sequence for $\| . \|_C$ and x is a τ-cluster point of (x_n), it is easily seen that $\| x - x_n \|_C$ tends to 0. It remains to apply the closed graph theorem to the canonical injection of $(E_C, \| . \|_C)$ into $(E, \| . \|)$.

Proof of (c):

Let L be the closure in (E, τ) of the set of the elements y of E satisfying $|y| \leq N \sum_0^N |x_n|$ for some integer N (L is the τ-closed order ideal of E generated by the x_n's). Let $\mathcal{T}_L$ be the linear topology on L which admits as a basis of zero-neighbourhoods the sets

$V_N(\in) = \{x \in L : \| |x| \wedge \sum_{n=0}^N |x_n| \| \leq \varepsilon \}$, $\varepsilon > 0$, $N = 0,1\ldots\ldots$

This topology $\mathcal{T}_L$ on L is coarser than the topology induced by τ. Let us prove that $\mathcal{T}_L$ is Hausdroff (then $\mathcal{T}_L$ is metrizable, τ and $\mathcal{T}_L$ coincide on $K \cap L$ by compactness, and we are done). We have to show that x is null if $x \in L$ and $|x| \wedge \sum_{n=0}^N |x_n| = 0$ for every N. But $\{ y \in E : |x| \wedge |y| = 0\}$ is a τ closed order ideal of E (it is easily checked that the lattice operations of E are τ continuous), so it contains x, where $x = 0$.

Lemma 2:

Let (u_n) and (v_n) be sequences of E converging to some point $c \in E$ for the topology τ, with $\lim_n \| |u_n - c| \wedge |u_n - c| \| = 0$. Then, for every sequence (w_n) of E and for every $x \in E$, we have

$$2 \lim_n \sup \| w_n - c \| \leq \lim_n \sup \| w_n - x \| + \lim_n \sup \| w_n - u_n \| + \lim_n \sup \| w_n - v_n \|.$$

Proof:

For u and x in E, let

$S_u(x) = x^+ \wedge |u| - \bar{x}^- \wedge |u|$

Then, for u, v, x, y in E the following inequalities hold.

(1) $\| S_u(x) \| \leq \| x \| \wedge (k \| u \|)$,

(2) $|| x - S_u(x) || \le || x - u ||$,

(3) $|| S_u(x) - S_u(y) || \le 2k \; || \; |x - y| \wedge |u| \; ||$,

(4) $|| S_u(x) + S_v(x) || \le || x || + k \, || \; |u| \wedge |v| \; ||$.

We get (1) using (α) and (β) since $(Su(x))^+ \le x^+$, $(Su(x))^- \le x^-$ and $| Su(x) |^+ \le | u |$. We also have $(x - S_u(x))^+ = (x - | u |)^+ \le (x - u)^+$ and, changing x and u into their opposite $(x - S_u(x))^- \le (x - u)^-$; this gives (2). The inequality (3) is given by

$$| S_u(x) - S_u(y) | \le | x^+ \wedge | u | - y^+ \wedge | u || + | y^- \wedge | u | - x^- \wedge | u ||$$
$$\le | x^+ - y^+ | \wedge | u | + | y^- - x^- | \wedge | u | \le 2(| x - y | \wedge | u |)$$

We get (4) using (1) and equality $S_u(x) + S_v(x) = S_{|u| \vee |v|}(x) + S_{|u| \wedge |v|}(x)$, an easy consequence of the identity $a + b = a \vee b + a \wedge b$.

Now we prove lemma 2 as follows. Without loss of generality, we assume c = 0. Using (2), we have

$$2|| w_n || \le || S_{u_n}(w_n) + S_{v_n}(w_n) || + || w_n - S_{u_n}(w_n) || + || w_n - S_{v_n}(w_n) ||$$
$$\le || S_{u_n}(w_n) + S_{v_n}(w_n) || + || w_n - u_n || + || w_n - v_n ||$$

By (3) and since $\lim_n u_n = 0$ for the topology τ, we have

$$\lim_n \sup || S_{u_n}(w_n) - S_{u_n}(w_n - x) || \; 2^K \lim_n \sup || \; | x | \wedge | u_n | \; || = 0,$$

and similarly $|| S_{v_n}(w_n) - S_{v_n}(w_n - x) ||$ tends to 0. So, using (4)

$$\lim_n \sup || S_{u_n}(w_n) + S_{v_n}(w_n) || = \lim_n \sup || S_{u_n}(w_n - x) + S_{v_n}(w_n - x) ||$$
$$\le \lim_n \sup || w_n - x || + k \lim_n \sup || \; | u_n | \wedge | u_n | \; || + \lim_n \sup || w_n - x ||.$$

This achieves the proof of Lemma 2.

A sequence (z_n) of B is said to be an "approximate fixed point sequence" when it verifies

(5) $\lim_n || T(z_n) - z_n || = 0$

Lemma 3:

there exists in B a point c and an approximate fixed points sequence (z_n) satisfying

(6) $\lim_n \sup || z_n - c || \le \frac{1}{4} (k+2)$ Diam (B).

Where Diam (B) = sup {$|| x - y ||$: (x , y) ∈ B x B}.

Proof:

For u ∈ B and λ ∈ (0, 1) the mapping

$T_{u\lambda}(x) = (1 - \lambda)u + \lambda T(x), \quad x \in B$

satisfying $|| T_u\lambda(x) - T_u\lambda(y) || \le \lambda || x - y ||$ for x, y in B. So it has a unique fixed point u(λ) ∈ B. we have for u, v in B and λ ∈ (0, 1)

(7) $T(u(\lambda)) = (1 - \lambda^{-1}) u + \lambda^{-1} u(\lambda)$.

(8) $||u(\lambda) - u|| \le \lambda (1 - \lambda)^{-1} || T(u) - u ||$

(9) $||u(\lambda) - v(\lambda) || \le ||u - v ||$

The equality (7) is obvious. Since $T_{u\lambda}$ is λ – Lipschhitzian and $T_{u\lambda}(u(\lambda)) = u(\lambda)$, we have clearly $||u(\lambda) - x|| \le \lambda (1 - \lambda)^{-1} || T_{u\lambda}(x) - x ||$ for every x ∈B. whence (8) taking x = u. Finally, we deduce (9) from

$$||u(\lambda) - v(\lambda) || = || (1 - \lambda) (u - v) + (T(u(\lambda)) - T(v(\lambda)) ||$$
$$\le (1 - \lambda) ||u - v|| + \lambda ||u(\lambda) - (v(\lambda) ||$$

As it is well known, the u(λ)'s yield approximate fixed points sequences since $|| (T(u(\lambda)) - u(\lambda) || = || (1 - \lambda)(T(u(\lambda)) - u || \le (1 - \lambda)$ Diam (By, with Diam (B) < ∞ (Lemma 1 (b)). So, using Lemma 1(c), we can find in B an approximate fixed points sequence $(c_n)_{n \ge 0}$ τ-converging to some point c ∈ B. By definition of τ, we may even assume that $|| \; |c_{2n} - c | \wedge |c_{2n+1} - c | \; ||$ tends to 0. Hence, letting $u_n = c_{2n}$, $v_n = c_{2n+1}$, we have

(10) $\lim_n || T(u_n) - u_n || = \lim_n || T(v_n) - v_n || = 0$.

(11) $\tau - \lim_n u_n = \tau - \lim_n v_n = c$.

(12) $\lim_n || \; | u_n - c | \wedge | v_n - c | \; || = 0$.

Then the sequence $w_n = \frac{1}{2} (u_n + v_n)$, n = 0, 1,……. Satisfies

(13) $\lim_n \sup || w_n - c || \le \frac{k}{2}$ Diam (B).

Indeed, for x, y in E, we have

$$|| x + y || \le x + y - 2(x^+ \wedge y^+) + 2(x^- \wedge y^-) + 2|| x^+ \wedge y^+ - x^- \wedge y^- ||$$

$\leq k \,||\, x - y \,||\, + 2k \,||\, |x| \wedge |y| \,||$.

since $| x^+ \wedge y^+ - x^- \wedge y^- | \leq | x | \wedge | y |$ and the identity $a + b - 2(a \wedge b) = | a - b |$ gives $| x + y - 2(x^+ \wedge y^+) + 2(x^- \wedge y^-) | = || x^+ - y^+| - |x^- - y^-|| \leq | x - y |$. Applying this with $x = u_n - c$ and $y = u_n - c$ and using (12) we get (13).

Let us now pick some fixed point number $\lambda \in (0, 1)$. From (9) and (13) we deduce

(14) $\lim_n \sup || w_n (\lambda) - c (\lambda) || \leq \frac{k}{2}$ Diam (B).

Further more, we have

(15) $\lim_n \sup (|| w_n (\lambda) - u_n || \vee || w_n (\lambda) - v_n ||) \leq \frac{1}{2}$ Diam (B).

Indeed, by (8) and (10), $\lim_n || u_n (\lambda) - u_n || = 0$, so using (9),

$\lim_n \sup (|| w_n (\lambda) - u_n || \leq \lim_n \sup || w_n - u_n ||) \leq \frac{1}{2}$ Diam (B).

The same is true for the $|| w_n (\lambda) - v_n ||$, whence (15). In view of (11), (12), (14) and (15), Lemma (2) gives

$\lim_n \sup || w_n (\lambda) - c || \leq \frac{1}{4}$ (k +2) Diam (B).

Now, if $\lambda_n \in (0, 1)$, n =0, 1,……., with $\lim_n \lambda_n = 1$, there exists a sequence $z_n = w_{h_n}(\lambda_n)$ such that $\lim_n \sup || z_n - c || \leq \frac{1}{4}$ (k +2) Diam (B). by (7), (z_n) is an approximate fixed points sequence, this proves Lemma 3.

The following lemma is essentially due to Goebel (1975) and Karlovitz (1976). In Goebel (1975) and Karlovitz (1976) the topology α below is the weak topology.

Lemma 4 (Goebel and Karlovitz):

Let F be a Banach space and let K be a convex bounded nonvoid subset of F compact for some Hausdroff topology α. We assume that, for every sequence (x_n) of K, the map $r(c) = \lim \sup_n || x_n - c ||$, $c \in K$, is lowering semi-continuous on K for the topology α. Let $T : K \rightarrow K$ be a non-expensive mapping. We suppose that K is minimal, in the sense that every α-compact convex nonvoid T-invariant subset of K is equal to K. Then we have

$$\lim_n || x_n - c || = \text{Diam } (K)$$

for every approximate fixed points sequence (x_n) of K and every $c \in K$.

As the above mapping r(c) is weakly lower semi-continuous (it is convex and norm-continuous), Lemma 4 contains the statement of Goebel and Karlovitz.

Proof:

First we prove that we have, for every $x \in K$.

(16) $\sup \{|| x - c ||: c \in K\}$ Diam (K).

When α is the weak topology, this is a result of Kirk (1965), and we follow Kirk's proof. Of course the hypothesis on α implies that the map $x \rightarrow || x - c ||$, $x \in K$, is α-lower semi-continuous for every $c \in K$; hence so the map $d(x) = \sup_{c\in K} || x - c ||$. Therefore, if $m = \inf_{x\in K} d(x)$, the set $K_o = \{x \in K: d(x) = m\}$ is nonvoid α-compact convex subset of K. It remains to prove that K_o is T-invariant then K_o =K and m = Diam(K). So, let $x \in K_o$. The set $\{ c \in K : || T(x) - c || \leq m\}$ is an α-compact convex set which is nonvoid and T-invariant since it contains T(K) by non-expensiveness of T. Hence it is equal to K and $T(x) \in K_o$.

As a consequence of (16) we get $\sup \{r(c): c \in K\}$ = Diam(K) for every sequence (x_n) of K if $r(c) = \lim \sup || x_n - c ||$: indeed $r(c) \geq || x - c ||$ if x is an α-cluster point of (x_n), by α-lower semi-continuity. But, if (x_n) is an approximate fixed points sequence, the map r(c) is convex, α-lower semi-continuous and verifies $r(T(c)) \leq r(c)$. By minimality of K, it is constant on K. So r(c) = Diam (K) for every $c \in K$, which easily gives the lemma.

The following result shows that Lemma 4 can be applied to the space E of the theorem, with α = τ.

Lemma 5:

For every norm-bounded and relatively τ- compact sequence (x_n) of E, the mapping r(c): $\lim \sup || x_n - c ||$, $c \in E$, is lower semi-continuous on (E, τ).

Proof:

Let $c \in E$ and $\varepsilon > 0$. It suffices to show that $r(c+x) \geq r(c) - \in$ for every x in some τ-neighbourhood V of the origin. Using lemma 1 we have r(c): $\lim_n || y_n ||$ for some sub-sequence (y_n) of ($x_n - c$) converging to some y ∈ E for the topology τ. The set V = { x ∈ E : $|| \, |x| \wedge |y| \, || \le k^{-1}\frac{\varepsilon}{3}$} is a τ-neighbourhood of the origin. Let x ∈ V. Since $\lim_n || \, |y_n - y \wedge |x| \, || = 0$, we have

$$\lim_n \sup || \, |y_n| \wedge |x| \, || \le k^{-1}\frac{\varepsilon}{3}.$$

Using the functions S_u (cf. Lemma 2) this gives

$$\lim_n \sup || y_n - x || \ge \lim_n \sup || y_n - S_x(y_n) - (x - S_{y_n}(x)) || - \frac{2\varepsilon}{3}$$

Since $| S_x(y_n) | = |S_{y_n}(x)| = | y_n | \wedge | x |$. But $y_n - S_x(y_n)$ and $x - S_{y_n}(x)$ are easily seen to be disjoint. So (condition (α) of the theorem), we get

$$\lim_n \sup || y_n - x || \ge \lim_n \sup || y_n - S_x(y_n) || - \frac{2\varepsilon}{3}$$
$$\ge \lim_n || y_n || - \varepsilon$$

Whence r(c+x) ≥ r(c) – ε.

Proof of the theorem:

Using Zorn's lemma we may assume that the set B of the theorem is minimal (cf. lemma 4). Let us consider the approximate fixed points sequence $(z_n) \subset B$ and the point c ∈B given by Lemma 3. In view of Lemmas 4 and 3 and Lemma1(b) we have $\lim_n || z_n - c || = \mathrm{Diam}(B)$, whence Diam(B) = 0 if k < 2.

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