COMMON FIXED POINT THEOREMS AND NON-EXPANSIVE MAPPING IN BANACH SPACE

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ABSTRACT

In this paper we explain some common fixed point theorems and non-expansive mapping in Banach space. Our aim is to generalize the theorems and non-expansive mapping in Banach space.

Key words: Fixed point theorems, Non expansive mapping, Banach space.

INTRODUCTION

We establish a common fixed point theorem for self mappings, not necessarily commuting of a closed and convex subset of a Banach space, generalizing a well known result of Gregus (1980).

Let X be a Banach space and T be a mapping of X into itself satisfying the inequality $\| Tx - Ty \| \leq \| x - y \|$ for all x, y in X. T is said to be non-expensive and it is well known that the class of contraction mapping and it is properly contained in the class of all continuous mapping. Kirk (1965) has independently proved a fixed point theorem for non-expansive mappings defined on a closed, bounded and convex subset of a uniformly convex Banach space and in spaces with richer structure. A number of generalizations of non-expensive mappings have been discussed by many authors. The works of Dotson (1972a and b); Emmanuele (1981); Goebel (1969); Goebel and Zlotkiewicz (1971); Goebel, Kirk and Shimi (1973); Massa and Roux (1978), Rhoades (1982) are of special significance. A comprehensive survey concerning fixed point theorems for non-expansive and related mappings can be found in Kirk (1965, 1981, 1983).

On the other hand, there are mappings which satisfy conditions similar to non-expansive and which possess a unique fixed point.

But such mapping cannot be viewed as generalizations of non-expansive mappings. Two such examples occur recently in Gregus (1980) and Rhoades (1978).

Motivated by a contractive condition of Hardy and Rogers (1973) in this chapter we extend the result of Gregus (1980) to the case of two mappings.

Let C be a closed convex subset of X. By summary, assuming b = c in the contractive condition of Gregus (1980), this author proved the following result.

Theorem 1:

Let T be a mapping of C into itself satisfying the inequality

(1) $\| Tx - Ty \| \leq a. \| x - y \| + b . \{\| Tx - x \| + \| Ty - y \| \}$

for all x, y in C, where $0<a<1$, $b>0$ and $a+2b = 1$. Then T has a unique fixed point.

We now prove the following theorem.

Theorem 2:

Let S and T be mappings of C into itself satisfying the inequality

(2) $\| Sx - Ty \| \leq a. \| x - y \| + b. \{\| Sx - x \| + \| Ty - y \| \} + c. \{\| Sx - y \| + \| Ty - x \| \}$

for all x, y in C, where $0<a<1$, $b>0$ and $a+2b+2c = 1$ and $(1 - b). c < ab$. If

(3) $\| Tx - x \| \leq \| Sx - x \|$

for all x in C, then S and T have a unique common fixed point w in C. Further, w is the unique fixed point of S and T.

Proof:

Let x be an arbitrary point in C. From (2), we deduce that

$|| STx - Tx || \leq a. \ || Tx - x || + b. \{|| STx - Tx || + || Tx - x || \} + c. \{|| STx - Tx || + || Tx - x || \},$

which implies that

(4) $|| STx - Tx || \leq \frac{a+b+c}{1-b-c} . || Tx - x || = || Tx - x ||.$

Similarly, we have

(5) $|| TSx - Sx || \leq || Sx - x ||.$

Since (4) holds for all x in C, we deduce that

$|| STSx - STx || \leq || TSx - Sx ||,$

Which implies, by (3) and (5), that

(6) $|| TTSx - TSx || \leq || STSx - TSx || \leq || Sx - x ||.$

We now define the point z by

$$z = \frac{1}{2} TSx + \frac{1}{2} TTSx.$$

Then, it follows, from (6), that

(7) $2|| TSx - z || = 2 \ || TTSx - z || = || TTSx - TSx || \leq || Sx - x ||.$

Since C is convex, z belongs to C and using (2), (5), (6) and (7), we have that

(8)
$$\begin{aligned} 2|| Sz - z || &= || 2Sz - (TSx + TTSx) || = || Sz - TSx || + || Sz - TTSx || \\ &\leq || Sz - TSx || + || Sz - TTSx || \\ &\leq a . \ || z - Sx || + b . \{ || Sz - z || + || Sx - x || \} \\ &+ c . \ \{ || Sz - z || + || Sx - z || + || TSx - z || \} \\ &+ a . \ || z - TSx || + b . \{ || Sz - z || + || Sx - x || \} \\ &+ c . \ \{ || Sz - z || + || TSx - z || + || TTSx - z || \} \\ &\leq a . \ \{ || Sx - z || + \tfrac{1}{2} . || Sx - x ||\} + 2b . \{ || Sz - z || + || Sx - x ||\} \\ &+ c . \ \{ 2|| Sz - z || + || Sx - z || + \tfrac{3}{2} . || Sx - x ||\}. \end{aligned}$$

On the other hand, using (2), (5) and (6), we obtain that

(9)
$$\begin{aligned} 2|| Sx - z || &= || 2Sx - (TSx + TTSx) || = || Sx - TSx || + || Sx - TTSx || \\ &\leq || Sx - TSx || + || Sx - TTSx || \\ &\leq || Sx - x || + a . || x - TSx || + b . \{ || Sx - x || + || Sx - x || \} \\ &+ c . \ \{ || Sx - x || + || TTSx - TSx || + || TSx - Sx || + || Sx - x || \} \\ &\leq || Sx - x || + a . \{ || Sx - x || + || TSx - Sx || \} \\ &+ (2b + 4c) . || Sx - x || \\ &\leq (1 + 2a + 2b + 4c) . || Sx - x || \\ &= (3 - 2b) . || Sx - x ||. \end{aligned}$$

It is easily seen that (8) and (9) imply that

$$2|| Sz - z || \leq a . (2 - b) . || Sx - x || + 2b . \{|| Sx - x || + || Sz - z ||\} + c . \{2|| Sz - z || + (3 - b) . || Sx - x ||\}.$$

Consequently we have that

(10) $|| Sz - z || \leq \lambda . || Sx - x ||,$

Where

$$\lambda = \frac{1}{2} \left(\frac{2a-ab+2b+3c-bc}{1-b-c}\right)$$

from the assumptions on the constants a,b and c, it follows that $0 < \underline{\lambda} < 1$. We claim that $h = \inf \{|| Sx - x || : x \in C\} = 0$, otherwise, for $0 < \underline{\varepsilon} < (1 - \underline{\lambda}) . h/\underline{\lambda}$, there exists a point $\bar{x}$ in C such that $|| S\bar{x} - \bar{x} || \leq h + \underline{\in}$ and hence (10) implies that $h \leq || Sz - z || \leq \underline{\lambda} . || S\bar{x} - \bar{x} || \leq \underline{\lambda} . (h + \underline{\in}) < h$, a contradiction.

Thus h=0 and the sets

$H_n = \{x \in C : \{|| Sx - x || \leq \frac{1}{n} \}$

are non-empty for any n = 1, 2, …; of course, we have

(11) $H_1 \supseteq H_2 \supseteq \ldots\ldots \supseteq H_n \supseteq \ldots\ldots$

Let $\overline{H}_n$ be the closure of H_n. We now show that

(12) diam $\overline{H}_n \leq (3 - a)/2bn$

for any n = 1, 2, ……. Indeed, we obtain on using (2) for all x, y in H_n,

$$\begin{aligned} \| x - y \| &\leq \| Sx - x \| + \| Sx - y \| \\ &\leq \| Sx - x \| + \| Ty - y \| + \| Sx - Ty \| \\ &\leq \tfrac{2}{n} + a\,.\, \| x - y \| + b\,.\, \{\| Sx - x \| + \| Ty - y \|\} \\ &\quad + c\,.\, \{\| Sx - x \| + \| x - y \| + \| Ty - y \| + \| x - y \|\} \\ &\leq \tfrac{2}{n} + (a + 2c)\,.\, \| x - y \| + (2b + 2c)/n \\ &= (3 - a)/ n + (1 - 2b). \| x - y \| \end{aligned}$$

Since (3) implies that $\| Ty - y \| \leq \| Sy - y \| \leq \frac{1}{n}$. The above inequality implies (12) since diam H_n = diam $\overline{H}_n$ and clearly it follows from (11) that

$\overline{H}_1 \supseteq \overline{H}_2 \supseteq \ldots\ldots \supseteq \overline{H}_n \supseteq \ldots\ldots$

Thus $\{\overline{H}_n\}$ is a decreasing sequence of non-empty subsets of C such that the sequence {diam $\overline{H}_n$} converges to zero as $n \to \infty$ by (12). Since X is complete, so is C and by Cantor's intersection theorem, there exists a point w in C such that

$w \in \bigcap_{n=1}^{\infty} \overline{H}_n$.

This means that $\| Sw - w \| \leq \frac{1}{n}$ for any n = 1, 2, …… and so Sw = w. Using (3), we have Tw = w. Then w is a common fixed point of S and T. Let us suppose that w' is another fixed point of S. On using (2) for x = w and y = w', we have that

$$\begin{aligned} \| w' - w \| &= \| Sw' - Tw \| \\ &\leq a\,.\, \| w' - w \| + c\,.\, \{\| w' - w \| + \| w - w' \|\} \\ &= (a + 2c)\,.\, \| w' - w \|. \end{aligned}$$

This implies that w' = w since a + 2c = 1 – 2b <1. Therefore w is the unique fixed point of S and similarly it is shown that w is the unique fixed point of T. This completes the proof.

Remark:

By assuming S=T and c=o, theorem 2 becomes theorem 1.

By enunciating theorem 2 for some iterates of S and T, we obtain the following result.

Theorem 3:

Let S and T be mapping of C into itself satisfying the inequality

$$\| S_{p_x} - T_{q_y} \| \leq a\,.\, \| x - y \| + b. \{\| S_{p_x} - x \| + \| T_{q_y} - y \|\}$$
$$+ c. \{\| S_{p_x} - y \| + \| T_{q_y} - x \|\}$$

for all x, y in C, where p and q are positive integers and a, b, c are as in theorem 2. If

$\| T_{q_y} - x \| \leq \| S_{p_x} - x \|$

for all x in C, then S and T have a unique common fixed point w in C. Further, w is the unique fixed point of S and T.

Proof:

By theorem 2, mapping Sp and Tq of C into itself have a unique common fixed point w in C. Since Sw = SSpw = SpSw, we deduce that Sw is also a fixed point of Sp, it follows that Sw = w. Similarly, we can prove that Tw = w and therefore w is common fixed point S and T. If w' is another fixed point of S, then we have that Spw' = w' but the uniqueness of w implies w = w'. Thus w is also the fixed point of S as well as for the mapping of T.

The following example shows the stronger generality of theorem 3 over theorem 2.

Example:

Let X be the Banach space of reals with Euclidean norm and C = [0,2]. We define S and T by putting Sx=0 if 0≤x<1, Sx = $\frac{3}{5}$ if 1≤x≤2, Tx=0 if 0≤x<2 and $T_2 = \frac{9}{5}$. Then the condition (2) of theorem 1 does not hold, otherwise, we should have for x =1 and y = 2.

$$\frac{6}{5} = \| S_1 - T_2 \| \leq a\,.\, \| 2 - 1 \| + b. \{\| 1 - \tfrac{3}{5} \| + \| 2 - \tfrac{9}{5} \|\}$$
$$+ c. \{\| \tfrac{9}{5} - 1 \| + \| 2 - \tfrac{3}{5} \|\}$$
$$= a + \frac{3b}{5} + \frac{11c}{5}$$

$= 1 - 2b - 2c + \frac{3b}{5} + \frac{11c}{5}$

Which implies $\frac{1}{5} + \frac{7b}{5} \leq \frac{c}{5}$, i.e, 1 + 7b ≤ c, a contradiction. However, the conditions of theorem 3 are trivially satisfied for p=q=2 since $S^2x=T^2x=0$ for all x in C.

We explicitly observe that the results of this chapter, for S = T, are not comparable with the results, where, although the contradictive condition used in more general than (2), the additional assumptions on the coefficients and the uniform convexity of X neither imply nor are implied by the assumptions of theorem 2.

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