Zero's of a polynomial in a prescribed region with restricted complex coefficients

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Abstract: The well known result in the theory of the distribution of zero's of polynomial due to Enestrom – Kakeya states that if $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree *n*, satisfying the condition

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$$

Then all zero's of Polynomial $P(z)$ lie in $|z| \leq 1$.

In this paper we present further generalization of the above result by considering polynomials with complex coefficients and locate the regions which contain the zero's of a polynomial by putting restrictions on the coefficients.

Keywords: zero's of polynomial, regions, coefficients.

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1. Introduction

The following famous result due to Enestrom – Kakeya [5] deals with the location of zero's of a polynomial with monotonic increasing positive co-efficients.

Theorem A: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree *n*, with real coefficients, satisfying

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$$

Then all zero's of $P(z)$ lie in $|z| \leq 1$.

Many researchers attempted to extend and generalize the Enestrom – Kakeya theorem by relaxing the hypothesis in several ways. An immediate generalization of Enestrom – Kakeya theorem is given by Joyal et al [4], infact they relaxed the non – negative condition on the coefficients of a polynomial and they proved the following result.

Theorem B. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree *n*, Satisfying

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0.$$

Then all zero's of $P(z)$ lie in $|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$.

Regarding the number of zero's of $P(z)$ in the region $|z| \leq \frac{1}{2}$, K.K Dewan [1], Proved the following result to the polynomials having complex coefficients.

Theorem C: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree *n*, with complex coefficients. Such that $\left|arg a_j - \beta\right| \leq \frac{\pi}{2},\ j = 0, 1, \ldots n$ for some real β *and*

$$|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then the number of zero's of $P(z)$ in $|z| < \frac{1}{2}$, does not exceed

$$\frac{1}{\log 2}\log\frac{|a_n|(\cos\alpha+\sin\alpha)+2\sin\alpha\sum_{j=0}^{n}|a_j|}{|a_0|}$$

Recently M. H Gulzar [2] , relaxed the hypothesis in several ways and in fact proved the following results.

Theorem D: Let $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ is a polynomial of degree n, with complex coefficients, such that $|arg\,a_j-\beta|\le\frac{\pi}{2},\ j=0,1,\dots n$ for some real β *and*

$$k|a_n|\ge|a_{n-1}|\ge\cdots\ge|a_1|\ge t|a_0|,$$

for $k\ge1, 0<t<1$, then all zero's of $P(z)$ in $\left|\frac{a_0}{M_3}\right|\le|z|\le\delta,\ \ 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k|a_n|(\cos\alpha+\sin\alpha+1)+2\sum_{j=1}^{n-1}|a_j|+2|a_0|-t|a_0|(\cos\alpha-\sin\alpha+1)}{|a_0|}$$

where $M_3=k|a_n|(\cos\alpha+\sin\alpha+1)-|a_0|(t\cos\alpha-t\sin\alpha-1+t)+2\sin\alpha\sum_{j=1}^{n-1}|a_j|$

Theorem E: Let $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ is a polynomial of degree n, with Complex coefficients, if $R_e(a_j)=\alpha_j$, $I_m(a_j)=\beta_j, j=0,1,2,\dots\dots n$, and

$$k\alpha_n\ge k\alpha_{n-1}\ge\cdots\ge\alpha_1\ge t\alpha_0,$$

$k\ge1, 0<t\le1$, then the number of zero's of $P(z)$ in $\left|\frac{a_0}{M_4}\right|\le|z|\le\delta,\ \ 0<\delta<1,$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k(|\alpha_n|+\alpha_n)+2|\alpha_0|-t(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}$$

where $M_4=k(|\alpha_n|+\alpha_n)+|\alpha_0|+|\beta_0|-t(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|$

2. Main results

In this paper, we generalize Enestrom – Kakeya by relaxing the hypothesis of a polynomial, in fact we prove the following.

Theorem 1: Let $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ be a polynomial of degree n, with Complex coefficients , Such that $|arg\ a_j-\beta|\le\alpha\le\frac{\pi}{2},\ j=0,1,\dots n$ for some real β *and*

$$k\,|a_n|\ge|a_{n-1}|\ge\cdots\ge|a_{p+1}|\ge|a_p|\le|a_{p-1}|\le\cdots\le|a_{q+1}|\le|a_q|\ge|a_{q-1}|\ge\cdots\ge|a_1|+\rho|a_0|$$

For $k\ge1, 0<\rho\le1$, then all the zero's of $P(z)$ in $\left|\frac{a_0}{M'}\right|\le|z|\le\delta,\ \ 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k|a_n|(\cos\alpha+\sin\alpha+1)+2\cos\alpha(|a_q|-|a_p|)+2|a_0|-\rho|a_0|(\cos\alpha-\sin\alpha+1)+S}{|a_0|}$$

Where $S = 2sin\alpha \sum_{j=1}^{n-1} |\alpha_j|$, and

$$M' = k|a_n|(cos\alpha + sin\,\alpha + 1) + 2cos\alpha(|a_q| - |a_p|) - |a_0|(\rho cos\alpha - \rho sin\alpha - 1 + \rho) + 2sin\alpha \sum_{j=1}^{n-1} |\alpha_j|$$

Remark: For $p = q = n$ theorem 1 reduces to theorem D.

Next we prove the following .

Theorem 2: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n, with Complex coefficients, if $R_e(a_j) = \alpha_j$, $I_m(a_j) = \beta_j$ $j = 0, 1, 2, \dots \dots . n$, and

$$k\alpha_n \geq k\alpha_{n-1} \geq \cdots \geq \alpha_{p+1} \geq \alpha_p \leq \alpha_{p-1} \leq \cdots \leq \alpha_{q+1} \leq \alpha_q \geq \alpha_{q-1} \geq \cdots \geq \alpha_1 \geq \rho\alpha_0 \,,$$

$k \geq 1$, $0 < \rho \leq 1$ then the number of zero's of $P(z)$ in $\left|\frac{a_0}{M''}\right| \leq |z| \leq \delta$, $0 < \delta < 1$ does not exceed

$$\frac{1}{log\frac{1}{\delta}} log \frac{k(|\alpha_n| + \alpha_n) + 2(\alpha_q - \alpha_p) - \rho(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}$$

Where $M'' = k(|\alpha_n| + \alpha_n) + |\alpha_0| + |\beta_0| + 2(\alpha_q - \alpha_p) - \rho(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=1}^{n} |\beta_j|$

Remark: for $p = q = n$ theorem 2 reduces to theorem E.

3. Proof of theorems:

For the proofs of the theorems, we need following lemmas.

Lemma 1: : Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n, with Complex coefficients , . Such that $|arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $j = 0, 1, \dots n$ for some real β, then for some $t > 0$

$$|ta_j - a_{j-1}| \leq [t|a_j| - |a_{j-1}|]cos\alpha + [t|a_j| + |a_{j-1}|]sin\alpha$$

Lemma 1 is due to Govel and Rahman [3]

Lemma 2: If $f(z)$ be analytic, $f(0) \neq 0$ and $|f(z)| \leq M$ in $|z| \leq 1$, then the number of zero's of $f(z)$ in $|z| \leq \delta$, $0 < \delta < 1$, does not exceed.

$$\frac{1}{log\frac{1}{\delta}} log \frac{M}{|f(0)|}$$

Lemma 2 is due to E.C Titchmarsh (See [6], Page 171)

Proof 1: Consider the Polynomial

$$F(z) = (1 - z)P(z)$$

$$= (1 - z)(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_{p+1} z^{p+1} + a_p z^p + \cdots + a_{q+1} z^{q+1} + a_q z^q + \dots + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \ldots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} - ka_n z^n + a_n z^n + (ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \cdots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - \rho a_0)z + \rho a_0 z - a_0 z + a_0$$

$$= -a_n z^{n+1} - (k-1)a_n z^n + (ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \cdots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - \rho a_0)z + (\rho - 1)a_0 z + a_0$$

$$|F(z)| = \left| -a_n z^{n+1} - (k-1)a_n z^n + (ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \cdots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - \rho a_0)z + (\rho - 1)\alpha_0 z + \alpha_0 \right|$$

For $|z| \le 1$, we have

$$|F(z)| \le |a_n| + (k-1)|a_n| + |ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{p+1} - a_p| + |a_p - a_{p-1}| + \cdots + |a_{q+1} - a_q| + |a_q - a_{q-1}| + \cdots + |a_1 - \rho a_0| + (1-\rho)|a_0| + |a_0|$$

$$\le |a_n| + (k-1)|a_n| + [(k|a_n| - |a_{n-1}|)\cos\alpha + (k|a_n| + |a_{n-1}|)\sin\alpha] + [(|a_{n-1}| - |a_{n-2}|)\cos\alpha + (|a_{n-1}| + |a_{n-2}|)\sin\alpha] + \cdots + [(|a_{p+1}| - |a_p|)\cos\alpha + (|a_{p+1}| + |a_p|)\sin\alpha] + [(|a_{p-1}| - |a_p|)\cos\alpha + (|a_{p-1}| + |a_p|)\sin\alpha] + \cdots + [(|a_q| - |a_{q+1}|)\cos\alpha + (|a_q| + |a_{q+1}|)\sin\alpha] + [(|a_q| - |a_{q-1}|)\cos\alpha + (|a_q| + |a_{q-1}|)\sin\alpha] + \cdots + [(|a_1| - \rho|a_0|)\cos\alpha + (|a_1| + \rho|a_0|)\sin\alpha] + (1-\rho)|a_0| + |a_0| + 2\sin\alpha \sum_{j=1}^{n-1} |a_j|$$

$$\le k|a_n|(\cos\alpha + \sin\alpha + 1) + 2\cos\alpha(|a_q| - |a_p|) + 2|a_0| - \rho|a_0|(\cos\alpha - \sin\alpha + 1) + S$$

Where $S = 2\sin\alpha \sum_{j=1}^{n-1} |a_j|$

Applying Lemma 2 to $F(z)$, it follows that number of zero's of $F(z)$ and hence of $P(z)$ in $|z| \le \delta$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{k|a_n|(\cos\alpha + \sin\alpha + 1) + 2\cos\alpha(|a_q| - |a_p| + 2|a_0| - \rho|a_0|(\cos\alpha - \sin\alpha + 1) + S}{|\alpha_0|}$$

Now to prove that $P(z)$ has no zero in $|z| < \frac{|\alpha_0|}{M'}$, we consider

$$F(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \ldots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - a_0)z + a_0$$

$$= a_0 + g(z)$$

Where

$$g(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \ldots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} - (k-1)a_n z^n + (ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \ldots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - \rho a_0)z + \rho a_0 z - a_0 z$$

For $|z| \le 1$, we have

$$|g(z)| \le |a_n| + (k-1)|a_n| + |ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{p+1} - a_p| + |a_p - a_{p-1}| + \cdots + |a_{q+1} - a_q| + |a_q - a_{q-1}| + \cdots + |a_1 - \rho a_0| + |\rho a_0 - a_0|$$

Using Lemma 1, we get

$$|g(z)| \le |a_n| + (k-1)|a_n| + [(k|a_n| - |a_{n-1}|)\cos\alpha + (k|a_n| + |a_{n-1}|)\sin\alpha] + [(|a_{n-1}| - |a_{n-2}|)\cos\alpha + (|a_{n-1}| + |a_{n-2}|)\sin\alpha] + \cdots + [(|a_{p+1}| - |a_p|)\cos\alpha + (|a_{p+1}| + |a_p|)\sin\alpha] + [(|a_{p-1}| - |a_p|)\cos\alpha + (|a_{p-1}| + |a_p|)\sin\alpha] + [(|a_q| - |a_{q+1}|)\cos\alpha + (|a_q| + |a_{q+1}|)\sin\alpha] + [(|a_q| - |a_{q-1}|)\cos\alpha + (|a_q| + |a_{q-1}|)\sin\alpha] + \cdots + [(|a_1| - \rho|a_0|)\cos\alpha + (|a_1| - \rho|a_0|)\sin\alpha] + (1-\rho)|a_0|$$

$$\le k|a_n|(\cos\alpha + \sin\alpha + 1) + 2\cos\alpha(|a_q| - |a_p| - |a_0|(\rho\cos\alpha - \rho\sin\alpha - 1 + \rho) + 2\sin\alpha \sum_{j=1}^{n-1} |a_j|$$

$|g(z)| \leq M'$

Where

$$M' = k|a_n|(cos\alpha + sin\alpha + 1) + 2cos\alpha(|a_q| - |a_p| - |a_0|(\rho cos\alpha - \rho sin\alpha - 1 + \rho)$$

$$+2sin\alpha \sum_{j=1}^{n-1} |a_j|$$

Therefore

$$|F(z)| = |a_0 + g(z)|$$

$$\geq |a_0| - |g(z)|$$

$$\geq |a_0| - |z||g(z)|$$

$$\geq |a_0| - |z|M' > 0$$

If $|z| < \frac{|a_0|}{M'}$

Hence $F(z)$ and therefore $P(z)$ has no zero in $|z| < \frac{|a_0|}{M'}$

Proof 2: Consider the Polynomial

$$F(z) = (1-z)P(z)$$

$$= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_{p+1}z^{p+1} + a_p z^p + a_{p-1}z^{p-1} + \cdots a_{q+1}z^{q+1} + a_q z^q + \dots + a_1 z + a_0)$$

$$= -(\alpha_n + \iota\beta_n)z^{n+1} + \alpha_n z^n - k\alpha_n z^n + (k\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \cdots + (\alpha_{p+1} - \alpha_p)z^{p+1} + (\alpha_p - \alpha_{p-1})z^p + \cdots + (\alpha_{q+1} - \alpha_q)z^{q+1} + (\alpha_q - \alpha_{q-1})z^q + \cdots + (\alpha_1 - \rho\alpha_0)z + (\rho\alpha_0 - \alpha_0) + \alpha_0 + \iota\left\{\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j + \beta_0\right\}$$

$$|F(z)| = \left|-(\alpha_n + \iota\beta_n)z^{n+1} + (\alpha_n - k\alpha_n)z^n + (k\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \cdots + (\alpha_{p+1} - \alpha_p)z^{p+1} + (\alpha_p - \alpha_{p-1})z^p + \cdots + (\alpha_{q+1} - \alpha_q)z^{q+1} + (\alpha_q - \alpha_{q-1})z^q + \cdots + (\alpha_1 - \rho\alpha_0)z + (1-\rho)\alpha_0 z + \alpha_0 + \iota\left\{\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j + \beta_0\right\}\right|$$

For $|z| \leq 1$, we have

$$|F(z)| \leq (|\alpha_n| + |\beta_n|) + |\alpha_n - k\alpha_n| + |k\alpha_n - k\alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + \cdots + |\alpha_{p+1} - \alpha_p| + |\alpha_p - \alpha_{p-1}| + \cdots + |\alpha_{q+1} - \alpha_q| + |\alpha_q - \alpha_{q-1}| + \cdots + |\alpha_1 - \rho\alpha_0| + (1-\rho)|\alpha_0| + |\alpha_0| + \sum_{j=0}^{n}(|\beta_j| + |\beta_{j-1}|) + |\beta_0|$$

$$\leq (|\alpha_n| + |\beta_n|) + (k-1)|\alpha_n| + k\alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \cdots + \alpha_{p+1} - \alpha_p + \alpha_{p-1} - \alpha_p + \cdots + \alpha_q - \alpha_{q+1} + \alpha_q - \alpha_{q-1} + \cdots + \alpha_1 - \rho\alpha_0 + (1-\rho)|\alpha_0| + |\alpha_0| + \sum_{j=0}^{n}(|\beta_j| + |\beta_{j-1}|) + |\beta_0|$$

$$\leq k|\alpha_n| + k\alpha_n - 2\alpha_p + 2\alpha_q - \rho\alpha_0 + (1-\rho)|\alpha_0| + |\alpha_0| + \sum_{j=0}^{n}(|\beta_j| + |\beta_{j-1}|) + |\beta_0|$$

$$\leq k(|\alpha_n| + \alpha_n) + 2(\alpha_q - \alpha_p) - \rho(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|$$

Therefore by applying Lemma 2 to $F(z)$, it follows that number of zero's of $F(z)$ and hence of $P(z)$ in $|z| \leq \delta$, does not exceed.

$$\frac{1}{log\frac{1}{\delta}} log\frac{k(|\alpha_n| + \alpha_n) + 2(\alpha_q - \alpha_p) - \rho(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}$$

Now to prove that $P(z)$ has no zero in $|z| < \frac{|\alpha_0|}{M''}$, we consider

$$F(z) = -(\alpha_n + \iota\beta_n)z^{n+1} + (1-k)\alpha_n z^n + (k\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \cdots + (\alpha_{p+1} - \alpha_p)z^{p+1} + (\alpha_p - \alpha_{p-1})z^p + \cdots + (\alpha_{q+1} - \alpha_q)z^{q+1} + (\alpha_q - \alpha_{q-1})z^q + \cdots + (\alpha_1 - \rho\alpha_0)z + (\rho - 1)\alpha_0 z + \iota\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j + a_0$$

$$= a_0 + g(z)$$

Where

$$g(z) = -(\alpha_n + \iota\beta_n)z^{n+1} + (1-k)\alpha_n z^n + (k\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \cdots + (\alpha_{p+1} - \alpha_p)z^{p+1} + (\alpha_p - \alpha_{p-1})z^p + \cdots + (\alpha_{q+1} - \alpha_q)z^{q+1} + (\alpha_q - \alpha_{q-1})z^q + \cdots + (\alpha_1 - \rho\alpha_0)z + (\rho - 1)\alpha_0 z + \iota\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j$$

For $|z| \leq 1$, we have

$$|g(z)| \leq |\alpha_n| + |\beta_n| + (k-1)|\alpha_n| + k\alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \cdots + \alpha_{p+1} - \alpha_p + \alpha_{p-1} - \alpha_p + \cdots + \alpha_q - \alpha_{q+1} + \alpha_q - \alpha_{q-1} + \cdots + \alpha_1 - \rho\alpha_0 + (1-\rho)|\alpha_0| + |\beta_1| + |\beta_0| + |\beta_2| + |\beta_1| + \cdots + |\beta_{n-1}| + |\beta_{n-2}| + |\beta_n| + |\beta_{n-1}|$$

$$\leq k|\alpha_n| + k\alpha_n + 2(\alpha_q - \alpha_p) - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=1}^{n} |\beta_j| + |\alpha_0| + |\beta_0|$$

$$\leq k(|\alpha_n| + \alpha_n) + 2(\alpha_q - \alpha_p) - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=1}^{n} |\beta_j| + |\alpha_0| + |\beta_0|$$

$$|g(z)| \leq M''$$

Where $M'' = k(|\alpha_n| + \alpha_n) + |\alpha_0| + |\beta_0| + 2(\alpha_q - \alpha_p) - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=1}^{n} |\beta_j|$

Therefore

$$|F(z)| = |a_0 + q(z)|$$

$$\geq |a_0| - |q(z)|$$

$$\geq |a_0| - |z||q(z)|$$

$$\geq |a_0| - |z|M'' > 0$$

If $|a_0| - |z|M'' > 0$

$$|a_0| > |z|M''$$

i.e $|z| < \frac{|a_0|}{M''}$

$$|z| < \frac{|a_0|}{k(|\alpha_n| + \alpha_n) + |\alpha_0| + |\beta_0| + 2(\alpha_q - \alpha_p) - \rho(|\alpha_0| + \alpha_0) + 2\sum_{j=1}^{n} |\beta_j|}$$

Hence $F(z)$ and $P(z)$ has no zero in $|z| < \frac{|a_0|}{M''}$.

References

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