FIRST ORDER DIFFERENTIAL SUBORDINATION OF ANALYTIC MAPS

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Abstract: In this work, we study about the certain form of differential subordinations and obtain the main results as well as applications of a first order differential subordination and starlikeness of analytic maps in the unit disc.

IndexTerms **- analytic functions, dominant functions, differential subordination.**

I. INTRODUCTION

Let H denote the space of analytic functions in the unit disc $U = \{z \in R : |z| < 1\}$, with the topology of local uniform convergence. Denote by A and A', the subspaces of H consisting of functions f which are normalized by the conditions $f(0) = f'(0) - 1 = 0$ and $f(0) = 1$, respectively. An analytic function p is said to satisfy the first order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z), h(0) = \psi(p(0), 0), z \in U. \quad (1.1)$$

A univalent function q is said to be the dominant of the differential subordination (1.1), if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ of (1.1) which satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1).

II. FIRST ORDER DIFFERENTIAL SUBORDINATION AND ITS DOMINANTS

LEMMA 2.1

Let F be analytic in U and let G be analytic and univalent in $\bar{U}$ except for the points ζ_0 such that $\lim_{z \to \zeta_0} F(z) = \infty$, with $F(0) = G(0)$. If F is not subordinate to G in U, then there is a point $z_0 \in U, \zeta_0 \in \partial U$ (boundary of U) such that $F(|z| < |z_0|) \subset G(U)$, $F(z_0) = G(\zeta_0)$ and $z_0 F'(z_0) = m\zeta_0 G'(\zeta_0)$ for some $m \geq 1$.

LEMMA 2.2

Let a complex number be α with $R\ \alpha > 0$. Suppose that $q \in A'$ is a convex univalent function which satisfies the following conditions:

(a) $R\ q(z) > 0, z \in \mathrm{U}$ when $R\ \alpha \geq |\alpha|^2$;

(b) $R\ q(z) > \dfrac{|\alpha|^2 - R\alpha}{2|\alpha|^2}, z \in U$, when $R\ \alpha < |\alpha|^2$.

If a function $p \in A'$ satisfies the differential subordination

$$(1-\alpha)p(z) + \alpha(p(z))^2 + \alpha z p'(z) \prec (1-\alpha)q(z) + \alpha(q(z))^2 + \alpha z q'(z) \quad (2.2.1)$$

in U and q is the best dominant.

PROOF

Define $h(z)$, where

$$h(z)=(1-\alpha)q(z)+\alpha(q(z))^2+\alpha zq'(z). \tag{2.2.2}$$

Clearly h is analytic in U and $h(0)=1.$

First of all, we will prove that h is univalent in U so that the subordination (2.2.1) is well defined in the unit disc U.

From "Equation (2.2.2)", we get

$$\frac{1}{\alpha}\frac{h'(z)}{q'(z)}=2q(z)+\frac{\bar{\alpha}-|\alpha|^2}{|\alpha|^2}+1+\frac{zq''(z)}{q'(z)}.$$

In view of the conditions (a) and (b) in Lemma 2.2 and the fact that q is convex in U, we obtain

$$R\ \frac{1}{\alpha}\frac{h'(z)}{q'(z)}>0, z\in U.$$

Since $R\ \alpha>0,$ and q is convex univalent in U, we finalize that h is close-to-convex and hence, univalent in U. We now show that $p\prec q.$

Without loss of generality, we can assume q to be analytic and univalent in $\bar{U}$.

If possible, suppose that $p\not\prec q$ in U.

Then, by Lemma 2.1,

There exists points $z_0\in U$ and $\zeta_0\in\partial U$ such that $p(z_0)=q(\zeta_0)$ and $z_0p'(z_0)=m\zeta_0q'(\zeta_0), m\geq 1.$

Thus,

$$(1-\alpha)p(z_0)+\alpha(p(z_0))^2+\alpha z_0p'(z_0)=(1-\alpha)q(\zeta_0)+\alpha(q(\zeta_0))^2+m\alpha\zeta_0q'(\zeta_0). \tag{2.2.3}$$

Consider a function,

$$L(z,t)=(1-\alpha)q(z)+\alpha(q(z))^2+\alpha tzq'(z), \tag{2.2.4}$$
$$=1+a_1(t)z+...$$

The function $L(z,t)$ is analytic in U for all $t\geq 0$ and is continuously differentiable on $[0,\infty)$ for all $z\in U.$ Now,

By "Equation (2.2.4)", we have

$$\frac{\partial L}{\partial z}=(1-\alpha)q'(z)+2\alpha q(z)q'(z)+\alpha tzq''(z)+\alpha tq'(z),$$

$$\left(\frac{\partial L}{\partial z}\right)_{(0,t)}=(1-\alpha)q'(0)+2\alpha q(0)q'(0)+\alpha t(0)q''(0)+\alpha tq'(0),$$

$$\left(\frac{\partial L}{\partial z}\right)_{(0,t)}=(1-\alpha)q'(0)+2\alpha(1)q'(0)+0+\alpha tq'(0),$$

$$\left(\frac{\partial L}{\partial z}\right)_{(0,t)}=q'(0)((1-\alpha)+2\alpha+\alpha t),$$

$$\left(\frac{\partial L}{\partial z}\right)_{(0,t)}=q'(0)(1-\alpha+2\alpha+\alpha t),$$

$$\left(\frac{\partial L}{\partial z}\right)_{(0,t)}=q'(0)(1+\alpha+\alpha t),$$

$$\therefore a_1(t)=\left[\frac{\partial L(z,t)}{\partial z}\right]_{(0,t)}=q'(0)(1+\alpha+\alpha t).$$

As the function q is univalent in U, therefore, $q'(0)\neq 0$. Also since $R\,\alpha>0$, we get $\left|\arg(1+\alpha+\alpha t)\right|<\frac{\pi}{2}$. Therefore, it follows that $a_1(t)\neq 0$ and $\lim_{t\to\infty}\left|a_1(t)\right|=\infty$.

Further, a simple calculation yields

By "Equation (2.2.4)",

$$\frac{\partial L}{\partial z}=(1-\alpha)q'(z)+2\alpha q(z)q'(z)+\alpha tzq''(z)+\alpha tq'(z),$$

$$z\frac{\partial L}{\partial z}=z(1-\alpha)q'(z)+2\alpha zq(z)q'(z)+\alpha tz^2q''(z)+\alpha ztq'(z),$$

$$\frac{\partial L}{\partial t}=\alpha zq'(z),$$

$$z\frac{\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}}=\frac{z(1-\alpha)q'(z)+2\alpha zq(z)q'(z)+\alpha tz^2q''(z)+\alpha ztq'(z)}{\alpha zq'(z)},$$

$$z\frac{\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}}=\frac{(1-\alpha)}{\alpha}+2q(z)+\frac{tzq''(z)}{q'(z)}+t,$$

$$z\frac{\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}}=2q(z)+\frac{(1-\alpha)}{\alpha}+t\left(1+\frac{zq''(z)}{q'(z)}\right),$$

thus,

$$z\frac{\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}}=2q(z)+\frac{\bar{\alpha}-|\alpha|^2}{|\alpha|^2}+t\left(1+\frac{zq''(z)}{q'(z)}\right),$$

clearly,

$$R\left[z\frac{\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}}\right]>0,\forall z\in U,t\geq 0.$$

In view of the given conditions (a) and (b) of Lemma 2.2, and the fact that q is convex in U. Hence, $L(z,t)$ is a subordination chain.

Therefore, $L(z,t_1)\prec L(z,t_2)$, for $0\leq t_1\leq t_2$.

From "Equation (2.2.4)",

We have $L(z,1)=h(z)$.

Thus we deduce that

$$L(\zeta_0,t)\not\in h(U) \text{ for } |\zeta_0|=1 \text{ and } t\geq 1. \qquad (2.2.5)$$

In view of "Equation (2.2.3) and (2.2.4)", we can write

$$(1-\alpha)p(z_0)+\alpha\left(p(z_0)\right)^2+\alpha z_0p'(z_0)=L(\zeta_0,m)\not\in h(U),$$

where $z_0\in U$, $|\zeta_0|=1$ and $m\geq 1$, which is a contradiction to (2.2.1).

Hence, $p\prec q$ in U.

THEOREM 2.3

Let α be a complex number with $R\ \alpha > 0$. For a function $g \in A$, set $G(z) = \frac{zg'(z)}{g(z)}$. Assume that $G(z)$ is a convex univalent function in U which satisfies the following conditions:

(a) $R\ G(z) > 0, z \in \mathrm{U}$ when $R\ \alpha \ge |\alpha|^2$;

(b) $R\ G(z) > \frac{|\alpha|^2 - R\alpha}{2|\alpha|^2}, z \in U$, when $R\ \alpha < |\alpha|^2$.

If a function $f \in A$ satisfies the differential subordination

$$\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \alpha \frac{z^2 g''(z)}{g(z)} + \frac{zg'(z)}{g(z)}, z \in U,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}, z \in U.$$

PROOF

The proof follows by setting $p(z) = \frac{zf'(z)}{f(z)}$ and $q(z) = \frac{zg'(z)}{g(z)}$ in Lemma 2.2.

We consider a first order differential subordination of the form

$$\frac{zf'(z)}{f(z)}\left(\alpha \frac{zf''(z)}{f'(z)} + 1\right) \prec \frac{zg'(z)}{g(z)}\left(\alpha \frac{zg''(z)}{g'(z)} + 1\right), z \in U,$$

where α is a complex number with $R\alpha > 0$.

$$\frac{\alpha zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}\right) + \frac{zf'(z)}{f(z)} \prec \frac{\alpha zg''(z)}{g'(z)}\left(\frac{zg'(z)}{g(z)}\right) + \frac{zg'(z)}{g(z)}, z \in U,$$

$$\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \frac{\alpha z^2 g''(z)}{g(z)} + \frac{zg'(z)}{g(z)}, z \in U.$$

By Lemma 2.2, we have

$p \prec q$ in U.

By setting $p(z) = \frac{zf'(z)}{f(z)}$ and $q(z) = \frac{zg'(z)}{g(z)}$ in above, we have

$$p(z) \prec q(z),$$

$$\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}, z \in U.$$

LEMMA 2.4

Let $G(z)$ be a convex univalent function which satisfies $RG(z) > 0$, for all $z \in U$. If $f \in A$ satisfies the differential subordination

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec (1-\alpha)G(z) + \alpha G^2(z) + \alpha z G'(z), z \in \mathrm{U},$$

then

$$\frac{zf'(z)}{f(z)} \prec G(z), z \in U.$$

PROOF

Taking α to be real such that $0<\alpha\le 1$ in Theorem 2.3, we obtain the following result

$$\alpha\frac{z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\prec\alpha\frac{z^2g''(z)}{g(z)}+\frac{zg'(z)}{g(z)},z\in U,$$

is the differential subordination.

Then

$$\frac{zf'(z)}{f(z)}\prec\frac{zg'(z)}{g(z)},z\in U.$$

Taking α to be real, such that $0<\alpha\le 1$

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2 f''(z)}{f(z)}\prec G(z).$$

Consider a function,

$$L(z,t)=(1-\alpha)q(z)+\alpha(q(z))^2+\alpha tzq'(z), \qquad (2.4.1)$$

$$=1+a_1(t)z+...$$

$$L(z,1)=h(z).$$

From "Equation (2.4.1)", we have

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2 f''(z)}{f(z)}\prec(1-\alpha)G(z)+\alpha G^2(z)+\alpha zG'(z),\mathrm{z}\in\mathrm{U},$$

then

$$\frac{zf'(z)}{f(z)}\prec G(z),z\in U.$$

THEOREM 2.5

Let $g\in A$ be a starlike function of order $\delta,\frac{1}{2}\le\delta\le 1$ and let $\frac{zg'(z)}{g(z)}=G(z)$. Assume that $G(z)$ is a convex univalent function in U. If $f\in A$ satisfies the differential subordination

$$\frac{z^2 f''(z)}{f(z)}\prec\frac{z^2g''(z)}{g(z)},\mathrm{z}\in\mathrm{U},$$

then

$$\frac{zf'(z)}{f(z)}\prec\frac{zg'(z)}{g(z)},z\in U.$$

PROOF

Assuming α to be real in the Theorem 2.3, the limiting case when α tends to infinity, gives the following result.

If a function $f\in A$ satisfies the differential subordination

$$\alpha\frac{z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\prec\alpha\frac{z^2g''(z)}{g(z)}+\frac{zg'(z)}{g(z)},z\in U, \qquad (2.5.1)$$

then

$$\frac{zf'(z)}{f(z)}\prec\frac{zg'(z)}{g(z)},z\in U.$$

when α approaches infinity in "Equation (2.5.1)", it becomes

$$\frac{z^2 f''(z)}{f(z)} \prec \frac{z^2 g''(z)}{g(z)}, z \in \mathrm{U},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}, z \in U.$$

III APPLICATIONS OF FIRST ORDER DIFFERENTIAL SUBORDINATION AND STARLIKENESS OF ANALYTIC MAPS IN THE UNIT DISC

LEMMA 3.1

Let $\alpha, \alpha \geq 0,$ be a real number. Assume that $a, |a| \leq 1,$ is a real number satisfying $a \leq \frac{1}{\alpha}$ whenever $\alpha > 1.$ Let $f \in A$ satisfy

$$\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec (1-\alpha)\left(\frac{1+az}{1-z}\right) + \alpha\left(\frac{1+az}{1-z}\right)^2 + \alpha\left(\frac{(a+1)z}{(1-z)^2}\right), z \in U,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1-z}, z \in U.$$

PROOF

If $f \in A$ satisfies the differential subordination

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec (1-\alpha)G(z) + \alpha G^2(z) + \alpha z G'(z), z \in \mathrm{U}, \qquad (3.1.1)$$

then

$$\frac{zf'(z)}{f(z)} \prec G(z), \forall z \in U.$$

Setting $G(z) = \frac{1+az}{1-z}, |a| \leq 1.$

$$G'(z) = \frac{(1-z)(a)-(1+az)(-1)}{(1-z)^2},$$

$$G'(z) = \frac{a(1-z)+1+az}{(1-z)^2},$$

$$G'(z) = \frac{a-az+1+az}{(1-z)^2},$$

$$G'(z) = \frac{1+a}{(1-z)^2}.$$

Then, from "Equation (3.1.1)",

$$\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec (1-\alpha)\left(\frac{1+az}{1-z}\right) + \alpha\left(\frac{1+az}{1-z}\right)^2 + \alpha\left(\frac{(a+1)z}{(1-z)^2}\right), z \in U,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1-z}, z \in U.$$

LEMMA 3.2

Let $\alpha, \alpha \geq 0$, be a real number. Assume that $\beta, 0 \leq \beta \leq 1$, is a real number which satisfies $\beta \geq \frac{1}{2} - \frac{1}{2\alpha}$ for $\alpha > 1$. If an analytic function f in A satisfies

$$R\left[\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right] > \alpha\beta^2 + \beta - \frac{\alpha}{2}(1+\beta), z \in U,$$

then $f \in S^*(\beta)$.

PROOF

From known result, we have
for $0 \leq \alpha, \beta \leq 1$, if $f \in A$ satisfies

$$R\left[\frac{zf'(z)}{f(z)}\left(\alpha \frac{zf''(z)}{f'(z)} + 1\right)\right] > \alpha\beta\left(\beta - \frac{1}{2}\right) + \beta - \frac{\alpha}{2}, z \in U,$$

then $f \in S^*(\beta)$.

$$R\left(\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > \alpha\beta\left(\beta - \frac{1}{2}\right) + \beta - \frac{\alpha}{2},$$

$$R\left(\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > \alpha\beta^2 - \frac{\alpha\beta}{2} + \beta - \frac{\alpha}{2},$$

$$R\left[\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right] > \alpha\beta^2 + \beta - \frac{\alpha}{2}(1+\beta), z \in U,$$

then $f \in S^*(\beta)$.

LEMMA 3.3

Let an analytic function f in A satisfy

$$R\frac{z^2 f''(z)}{f(z)} > \frac{2\beta^2 - \beta - 1}{2}, z \in U, \frac{1}{2} \leq \beta \leq 1.$$

Then $f \in S^*(\beta)$.

PROOF:

Let α tend to infinity in Lemma 3.2,

$$R\left[\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right] > \alpha\beta^2 + \beta - \frac{\alpha}{2}(1+\beta), z \in U,$$

$$R\left[\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right] > \frac{2\alpha\beta^2 + 2\beta - \alpha(1+\beta)}{2}, z \in U,$$

$$R\left[\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right] > \frac{2\alpha\beta^2 + 2\beta - \alpha - \alpha\beta}{2}, z \in U,$$

If α approaches infinity, then we get

$$R\left[\frac{z^2 f''(z)}{f(z)}\right] > \frac{2\beta^2 - \beta - 1}{2}, z \in U, \frac{1}{2} \leq \beta \leq 1,$$

then $f \in S^*(\beta)$.

LEMMA 3.4

Let α and β be real numbers such that $0<\alpha,\beta\le 1$, for all $z\in U$, let $f\in A$ satisfy

$$\frac{\alpha z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\prec(1-\alpha)\left(\frac{1+z}{1-z}\right)^{\beta}+\alpha\left(\frac{1+z}{1-z}\right)^{2\beta}+\beta\alpha\left(\frac{1+z}{1-z}\right)^{\beta-1}\left(\frac{2z}{(1-z)^2}\right),$$

then,

$$\frac{zf'(z)}{f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\beta},$$

in U.

$$i.e., f\in S(\beta).$$

PROOF:

If $f\in A$ satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2 f''(z)}{f(z)}\prec(1-\alpha)G(z)+\alpha G^2(z)+\alpha zG'(z), z\in U,$$

then,

$$\frac{zf'(z)}{f(z)}\prec G(z), \forall z\in U.$$

setting $G(z)=\left(\frac{1+z}{1-z}\right)^{\beta}, 0<\beta\le 1,$ we obtain

$$\frac{zf'(z)}{f(z)}+\frac{\alpha z^2 f''(z)}{f(z)}\prec(1-\alpha)\left(\frac{1+z}{1-z}\right)^{\beta}+\alpha\left(\frac{1+z}{1-z}\right)^{2\beta}+\alpha zG'(z), z\in U,$$

$$G'(z)=\left(\frac{1+z}{1-z}\right)^{\beta-1}\left(\frac{2\beta}{(1-z)^2}\right),$$

$$\frac{zf'(z)}{f(z)}+\frac{\alpha z^2 f''(z)}{f(z)}\prec(1-\alpha)\left(\frac{1+z}{1-z}\right)^{\beta}+\alpha\left(\frac{1+z}{1-z}\right)^{2\beta}+\alpha\beta\left(\frac{1+z}{1-z}\right)^{\beta-1}\left(\frac{2z}{(1-z)^2}\right), z\in U,$$

then,

$$\frac{zf'(z)}{f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\beta} \text{ in } U.$$

$i.e., f\in s(\beta).$

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