Weight Distribution of the k^{th} Power of the Sum of First *n* Natural Numbers

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Abstract:

In this paper an attempt has been made to explore the weight distribution of the k^{th} power of the sum of first n natural numbers; where k is a positive integer. At the end, a formula has been suggested and proved which is as follows:

The k^{th} power of the sum of the first n natural numbers is the Weighted Arithmetic Mean of the sum of their $(2k\text{-}1)^{th}$ powers, the sum of their $(2k\text{-}3)^{th}$ powers, the sum of their $(2k\text{-}5)^{th}$ powers,………with weights $\binom{k}{1}, \binom{k}{3}, \binom{k}{5}$,…...respectively ; where k is a positive integer.

Key words:

Series Summation ; Natural Numbers ; Series Summation of Natural Numbers ; Weight Distribution of Natural Numbers ; Finite Series of Natural Numbers

1. Introduction

The power series of first n natural numbers have been the area of interest to many eminent mathematicians as well as young scientists from time immemorial. One of the obvious reasons is many wonderful mathematical expressions and results have been derived based on that. For examples, already there exist beautiful formulae for the sum of first n natural numbers, sum of the squares of first *n* natural numbers, sum of the cubes of first *n* natural numbers,…etc.

Experiment plays a pivotal role in every discipline of Science and Mathematics is not the exception. Many interesting results in Mathematics were initially nothing but conjectures and eventually turned into well established formulae when they were successfully proved. In this paper, we are going to illustrate a result showing the weight distribution of the k^{th} power of the sum of first *n* natural numbers. Initiallywe make a conjecture and then finallywe prove the same using Binomial Theorem.

Let's do the following experiments and then observe the results:

Experiment(i):

$$(r+1)^1 r^1 - r^1 (r-1)^1$$

$$= r^1 \left\{ (r+1)^1 - (r-1)^1 \right\}$$

$$= r^1 \left[\left\{ \binom{1}{0} r^1 + \binom{1}{1} \right\} - \left\{ \binom{1}{0} r^1 - \binom{1}{1} \right\} \right]$$

$$= r^1 \left[2 \binom{1}{1} \right]$$

$$= 2\, r^1 \binom{1}{1}$$

$$\Rightarrow 2 \binom{1}{1} r^1 = (r+1)^1 r^1 - r^1 (r-1)^1 \qquad \textbf{(1)}$$

Putting $r = 1, 2, 3, \ldots\ldots, n$ in result **(1)** and adding, we get

$$2 \binom{1}{1} \left(\sum_{r=1}^{n} r^1 \right) = n^1 (n+1)^1$$

$$\Rightarrow \binom{1}{1} \left(\sum_{r=1}^{n} r^1 \right) = \frac{1}{2} \left\{ n(n+1) \right\}^1 \qquad \textbf{(2)}$$

Experiment (ii):

$$(r+1)^2 r^2 - r^2 (r-1)^2$$

$$= r^2 \left\{ (r+1)^2 - (r-1)^2 \right\}$$

$$= r^2 \left[\left\{ \binom{2}{0} r^2 + \binom{2}{1} r^1 + \binom{2}{2} \right\} - \left\{ \binom{2}{0} r^2 - \binom{2}{1} r^1 + \binom{2}{2} \right\} \right]$$

$$= r^2 \left[2 \binom{2}{1} r \right]$$

$$= 2\, r^3 \binom{2}{1}$$

$$\Rightarrow 2 \binom{2}{1} r^3 = (r+1)^2 r^2 - r^2 (r-1)^2 \qquad \textbf{(3)}$$

Putting $r = 1, 2, 3, \ldots\ldots, n$ in result **(3)** and adding, we get

$$2 \binom{2}{1} \left(\sum_{r=1}^{n} r^3 \right) = n^2 (n+1)^2$$

$$\Rightarrow \binom{2}{1} \left(\sum_{r=1}^{n} r^3 \right) = \frac{1}{2} \{ n(n+1) \}^2 \qquad \textbf{(4)}$$

Experiment (iii):

$$(r+1)^3 r^3 - r^3 (r-1)^3$$

$$= r^3 \left\{ (r+1)^3 - (r-1)^3 \right\}$$

$$= r^3 \left[\left\{ \binom{3}{0} r^3 + \binom{3}{1} r^2 + \binom{3}{2} r^1 + \binom{3}{3} \right\} - \left\{ \binom{3}{0} r^3 - \binom{3}{1} r^2 + \binom{3}{2} r^1 - \binom{3}{0} \right\} \right]$$

$$= r^3 \left[2 \left\{ \binom{3}{1} r^2 + \binom{3}{3} \right\} \right]$$

$$= 2 \left\{ \binom{3}{1} r^5 + \binom{3}{3} r^3 \right\}$$

$$\Rightarrow 2\left\{\binom{3}{1}r^5+\binom{3}{3}r^3\right\}=(r+1)^3r^3-r^3(r-1)^3 \quad \textbf{(5)}$$

Putting $r=1,2,3,\ldots\ldots,n$ in result **(5)** and adding, we get

$$2\left\{\binom{3}{1}\sum_{r=1}^{n}r^5+\binom{3}{3}\sum_{r=1}^{n}r^3\right\}=n^3(n+1)^3$$

$$\Rightarrow \binom{3}{1}\sum_{r=1}^{n}r^5+\binom{3}{3}\sum_{r=1}^{n}r^3=\frac{1}{2}\{n(n+1)\}^3 \quad \textbf{(6)}$$

Experiment (iv):

$$(r+1)^4r^4-r^4(r-1)^4$$

$$=r^4\left\{(r+1)^4-(r-1)^4\right\}$$

$$=r^4\left[\left\{\binom{4}{0}r^4+\binom{4}{1}r^3+\binom{4}{2}r^2+\binom{4}{3}r^1+\binom{4}{4}\right\}\right.$$
$$\left.-\left\{\binom{4}{0}r^4-\binom{4}{1}r^3+\binom{4}{2}r^2-\binom{4}{3}r^1+\binom{4}{4}\right\}\right]$$

$$=r^4\left[2\left\{\binom{4}{1}r^3+\binom{4}{3}r^1\right\}\right]$$

$$=2\left\{\binom{4}{1}r^7+\binom{4}{3}r^5\right\}$$

$$\Rightarrow 2\left\{\binom{4}{1}r^7+\binom{4}{3}r^5\right\}=(r+1)^4r^4-r^4(r-1)^4 \quad \textbf{(7)}$$

Putting $r=1,2,3,\ldots\ldots,n$ in result **(7)** and adding, we get

$$2\left\{\binom{4}{1}\sum_{r=1}^{n}r^7+\binom{4}{3}\sum_{r=1}^{n}r^5\right\}=n^4(n+1)^4$$

$$\Rightarrow \binom{4}{1}\sum_{r=1}^{n}r^7+\binom{4}{3}\sum_{r=1}^{n}r^5=\frac{1}{2}\{n(n+1)\}^4 \quad \textbf{(8)}$$

Experiment (v):

$$(r+1)^5 r^5 - r^5 (r-1)^5$$

$$= r^5 \left\{ (r+1)^5 - (r-1)^5 \right\}$$

$$= r^5 \left[\left\{ \binom{5}{0} r^5 + \binom{5}{1} r^4 + \binom{5}{2} r^3 + \binom{5}{3} r^2 + \binom{5}{4} r^1 + \binom{5}{5} \right\} - \left\{ \binom{5}{0} r^5 - \binom{5}{1} r^4 + \binom{5}{2} r^3 - \binom{5}{3} r^2 + \binom{5}{4} r^1 - \binom{5}{5} \right\} \right]$$

$$= r^5 \left[2 \left\{ \binom{5}{1} r^4 + \binom{5}{3} r^2 + \binom{5}{5} \right\} \right]$$

$$= 2 \left\{ \binom{5}{1} r^9 + \binom{5}{3} r^7 + \binom{5}{5} r^5 \right\}$$

$$\Rightarrow 2 \left\{ \binom{5}{1} r^9 + \binom{5}{3} r^7 + \binom{5}{5} r^5 \right\} = (r+1)^5 r^5 - r^5 (r-1)^5 \qquad \textbf{(9)}$$

Putting $r = 1, 2, 3, \ldots\ldots, n$ in result **(9)** and adding, we get

$$2 \left\{ \binom{5}{1} \sum_{r=1}^{n} r^9 + \binom{5}{3} \sum_{r=1}^{n} r^7 + \binom{5}{5} \sum_{r=1}^{n} r^5 \right\} = n^5 (n+1)^5$$

$$\Rightarrow \binom{5}{1} \sum_{r=1}^{n} r^9 + \binom{5}{3} \sum_{r=1}^{n} r^7 + \binom{5}{5} \sum_{r=1}^{n} r^5 = \frac{1}{2} \left\{ n(n+1) \right\}^5 \qquad \textbf{(10)}$$

Experiment (vi):

$$(r+1)^6 r^6 - r^6 (r-1)^6$$

$$= r^6 \left\{ (r+1)^6 - (r-1)^6 \right\}$$

$$= r^6 \left[\left\{ \binom{6}{0} r^6 + \binom{6}{1} r^5 + \binom{6}{2} r^4 + \binom{6}{3} r^3 + \binom{6}{4} r^2 + \binom{6}{5} r^1 + \binom{6}{6} \right\} - \left\{ \binom{6}{0} r^6 - \binom{6}{1} r^5 + \binom{6}{2} r^4 - \binom{6}{3} r^3 + \binom{6}{4} r^2 - \binom{6}{5} r^1 + \binom{6}{6} \right\} \right]$$

$$= r^6 \left[2 \left\{ \binom{6}{1} r^5 + \binom{6}{3} r^3 + \binom{6}{5} r^1 \right\} \right]$$

$$=2\left\{\binom{6}{1}r^{11}+\binom{6}{3}r^{9}+\binom{6}{5}r^{7}\right\}$$

$$\Rightarrow 2\left\{\binom{6}{1}r^{11}+\binom{6}{3}r^{9}+\binom{6}{5}r^{7}\right\}=(r+1)^6r^6-r^6(r-1)^6 \quad \textbf{(11)}$$

Putting $r=1,2,3,\ldots\ldots,n$ in result **(11)** and adding, we get

$$2\left\{\binom{6}{1}\sum_{r=1}^{n}r^{11}+\binom{6}{3}\sum_{r=1}^{n}r^{9}+\binom{6}{5}\sum_{r=1}^{n}r^{7}\right\}=n^6(n+1)^6$$

$$\Rightarrow \binom{6}{1}\sum_{r=1}^{n}r^{11}+\binom{6}{3}\sum_{r=1}^{n}r^{9}+\binom{6}{5}\sum_{r=1}^{n}r^{7}=\frac{1}{2}\{n(n+1)\}^6 \quad \textbf{(12)}$$

Experiment (vii):

$$(r+1)^7r^7-r^7(r-1)^7$$

$$=r^7\left\{(r+1)^7-(r-1)^7\right\}$$

$$=r^7\left[\left\{\binom{7}{0}r^7+\binom{7}{1}r^6+\binom{7}{2}r^5+\binom{7}{3}r^4+\binom{7}{4}r^3+\binom{7}{5}r^2+\binom{7}{6}r^1+\binom{7}{7}\right\}\right.$$
$$\left.-\left\{\binom{7}{0}r^7-\binom{7}{1}r^6+\binom{7}{2}r^5-\binom{7}{3}r^4+\binom{7}{4}r^3-\binom{7}{5}r^2+\binom{7}{6}r^1-\binom{7}{7}\right\}\right]$$

$$=r^7\left[2\left\{\binom{7}{1}r^6+\binom{7}{3}r^4+\binom{7}{5}r^2+\binom{7}{7}\right\}\right]$$

$$=2\left\{\binom{7}{1}r^{13}+\binom{7}{3}r^{11}+\binom{7}{5}r^{9}+\binom{7}{7}r^{7}\right\}$$

$$\Rightarrow 2\left\{\binom{7}{1}r^{13}+\binom{7}{3}r^{11}+\binom{7}{5}r^{9}+\binom{7}{7}r^{7}\right\}=(r+1)^7r^7-r^7(r-1)^7 \quad \textbf{(13)}$$

Putting $r=1,2,3,\ldots\ldots,n$ in result **(13)** and adding, we get

$$2\left\{\binom{7}{1}\sum_{r=1}^{n}r^{13}+\binom{7}{3}\sum_{r=1}^{n}r^{11}+\binom{7}{5}\sum_{r=1}^{n}r^{9}+\binom{7}{7}\sum_{r=1}^{n}r^{7}\right\}=n^7(n+1)^7$$

$$\Rightarrow \binom{7}{1}\sum_{r=1}^{n}r^{13}+\binom{7}{3}\sum_{r=1}^{n}r^{11}+\binom{7}{5}\sum_{r=1}^{n}r^{9}+\binom{7}{7}\sum_{r=1}^{n}r^{7}=\frac{1}{2}\{n(n+1)\}^7 \quad \textbf{(14)}$$

Experiment (viii):

$$(r+1)^8 r^8 - r^8 (r-1)^8$$

$$= r^8 \left\{ (r+1)^8 - (r-1)^8 \right\}$$

$$= r^8 \left[\left\{ \binom{8}{0} r^8 + \binom{8}{1} r^7 + \binom{8}{2} r^6 + \binom{8}{3} r^5 + \binom{8}{4} r^4 + \binom{8}{5} r^3 + \binom{8}{6} r^2 + \binom{8}{7} r^1 + \binom{8}{8} \right\} \right.$$
$$\left. - \left\{ \binom{8}{0} r^8 - \binom{8}{1} r^7 + \binom{8}{2} r^6 - \binom{8}{3} r^5 + \binom{8}{4} r^4 - \binom{8}{5} r^3 + \binom{8}{6} r^2 - \binom{8}{7} r^1 + \binom{8}{8} \right\} \right]$$

$$= r^8 \left[2 \left\{ \binom{8}{1} r^7 + \binom{8}{3} r^5 + \binom{8}{5} r^3 + \binom{8}{7} r^1 \right\} \right]$$

$$= 2 \left\{ \binom{8}{1} r^{15} + \binom{8}{3} r^{13} + \binom{8}{5} r^{11} + \binom{8}{7} r^9 \right\}$$

$$\Rightarrow 2 \left\{ \binom{8}{1} r^{15} + \binom{8}{3} r^{13} + \binom{8}{5} r^{11} + \binom{8}{7} r^9 \right\} = (r+1)^8 r^8 - r^8 (r-1)^8 \qquad \textbf{(15)}$$

Putting $r = 1, 2, 3, ..., n$ in result **(15)** and adding, we get

$$2 \left\{ \binom{8}{1} \sum_{r=1}^{n} r^{15} + \binom{8}{3} \sum_{r=1}^{n} r^{13} + \binom{8}{5} \sum_{r=1}^{n} r^{11} + \binom{8}{7} \sum_{r=1}^{n} r^9 \right\} = n^8 (n+1)^8$$

$$\Rightarrow \binom{8}{1} \sum_{r=1}^{n} r^{15} + \binom{8}{3} \sum_{r=1}^{n} r^{13} + \binom{8}{5} \sum_{r=1}^{n} r^{11} + \binom{8}{7} \sum_{r=1}^{n} r^9 = \frac{1}{2} \{ n(n+1) \}^8 \qquad \textbf{(16)}$$

2. Observation

We know that the sum of first n natural numbers is $\frac{n(n+1)}{2}$. So, in each of **(2)**,**(4)**,**(6)**, **(8)**,**(10)**,**(12)**, **(14)**and**(16)**, the term $n(n+1)$ can be rewritten as $2\left(\sum_{r=1}^{n} r\right)$. Accordingly, we get the following:

$$\binom{1}{1}\left(\sum_{r=1}^{n} r^{1}\right) = 2^{0}\left\{\sum_{r=1}^{n} r\right\}^{1} \qquad \textbf{(17)}$$

$$\binom{2}{1}\left(\sum_{r=1}^{n} r^{3}\right) = 2^{1}\left\{\sum_{r=1}^{n} r\right\}^{2} \qquad \textbf{(18)}$$

$$\binom{3}{1}\sum_{r=1}^{n} r^{5} + \binom{3}{3}\sum_{r=1}^{n} r^{3} = 2^{2}\left\{\sum_{r=1}^{n} r\right\}^{3} \qquad \textbf{(19)}$$

$$\binom{4}{1}\sum_{r=1}^{n} r^{7} + \binom{4}{3}\sum_{r=1}^{n} r^{5} = 2^{3}\left\{\sum_{r=1}^{n} r\right\}^{4} \qquad \textbf{(20)}$$

$$\binom{5}{1}\sum_{r=1}^{n} r^{9} + \binom{5}{3}\sum_{r=1}^{n} r^{7} + \binom{5}{5}\sum_{r=1}^{n} r^{5} = 2^{4}\left\{\sum_{r=1}^{n} r\right\}^{5} \qquad \textbf{(21)}$$

$$\binom{6}{1}\sum_{r=1}^{n} r^{11} + \binom{6}{3}\sum_{r=1}^{n} r^{9} + \binom{6}{5}\sum_{r=1}^{n} r^{7} = 2^{5}\left\{\sum_{r=1}^{n} r\right\}^{6} \qquad \textbf{(22)}$$

$$\binom{7}{1}\sum_{r=1}^{n} r^{13} + \binom{7}{3}\sum_{r=1}^{n} r^{11} + \binom{7}{5}\sum_{r=1}^{n} r^{9} + \binom{7}{7}\sum_{r=1}^{n} r^{7} = 2^{6}\left\{\sum_{r=1}^{n} r\right\}^{7} \qquad \textbf{(23)}$$

$$\binom{8}{1}\sum_{r=1}^{n} r^{15} + \binom{8}{3}\sum_{r=1}^{n} r^{13} + \binom{8}{5}\sum_{r=1}^{n} r^{11} + \binom{8}{7}\sum_{r=1}^{n} r^{9} = 2^{7}\left\{\sum_{r=1}^{n} r\right\}^{8} \qquad \textbf{(24)}$$

3. Conjecture

Based on**(17)**,**(18)**,**(19)**, **(20)**,**(21)**,**(22)**, **(23)**and**(24)**, we make a conjecture:

$$\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + \dots\dots = 2^{k-1}\left\{\sum_{r=1}^{n} r\right\}^{k} \quad \textbf{(25)}$$

Also we know that,

$$\binom{k}{1} + \binom{k}{3} + \binom{k}{5} + \dots\dots = 2^{k-1} \quad \textbf{(26)}$$

Dividing **(25)** by**(26)**,we get

$$\frac{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + \dots\dots}{\binom{k}{1} + \binom{k}{3} + \binom{k}{5} + \dots\dots} = \left\{\sum_{r=1}^{n} r\right\}^{k} \quad \textbf{(27)}$$

However, the above result obtained in **(27)** is merely a conjecture as it istrueonly for $k = 1, 2, 3, \dots\dots, 8$. Could it be true for all positive integral values of k? Could it be possible to convert this conjecture into an well established formula?

4. Formal Proof

From Binomial Theorem for a positive integral index k, we get

$$(r+1)^k = \binom{k}{0}r^k + \binom{k}{1}r^{k-1} + \binom{k}{2}r^{k-2} + \binom{k}{3}r^{k-3} + \dots\dots + \binom{k}{k} \quad \textbf{(28)}$$

and

$$(r-1)^k = \binom{k}{0}r^k - \binom{k}{1}r^{k-1} + \binom{k}{2}r^{k-2} - \binom{k}{3}r^{k-3} + \dots\dots + (-1)^k\binom{k}{k} \quad \textbf{(29)}$$

Subtracting **(29)** from **(28)**, we get

$$(r+1)^k - (r-1)^k = \left\{\binom{k}{0}r^k + \binom{k}{1}r^{k-1} + \binom{k}{2}r^{k-2} + \binom{k}{3}r^{k-3} + \dots\dots + \binom{k}{k}\right\}$$
$$-\left\{\binom{k}{0}r^k - \binom{k}{1}r^{k-1} + \binom{k}{2}r^{k-2} - \binom{k}{3}r^{k-3} + \dots\dots + (-1)^k\binom{k}{k}\right\} \quad \textbf{(30)}$$

Multiplying each side of **(30)** by r^k, we get

$$r^k\left\{(r+1)^k-(r-1)^k\right\}=r^k\left[\left\{\binom{k}{0}r^k+\binom{k}{1}r^{k-1}+\binom{k}{2}r^{k-2}+\binom{k}{3}r^{k-3}+.......+\binom{k}{k}\right\}\right.$$
$$\left.-\left\{\binom{k}{0}r^k-\binom{k}{1}r^{k-1}+\binom{k}{2}r^{k-2}-\binom{k}{3}r^{k-3}+.......+(-1)^k\binom{k}{k}\right\}\right]$$

$$\Rightarrow (r+1)^k r^k-r^k(r-1)^k=r^k\left[\left\{\binom{k}{0}r^k+\binom{k}{1}r^{k-1}+\binom{k}{2}r^{k-2}+\binom{k}{3}r^{k-3}+.......+\binom{k}{k}\right\}\right.$$
$$\left.-\left\{\binom{k}{0}r^k-\binom{k}{1}r^{k-1}+\binom{k}{2}r^{k-2}-\binom{k}{3}r^{k-3}+.......+(-1)^k\binom{k}{k}\right\}\right] \quad \textbf{(31)}$$

Now, we have two possibilities. One is when k is an odd positive integer whereas another is when k is an even positive integer. Let us prove both the possibilities one by one.

(i) When k is odd positive integer:

$$(r+1)^k r^k-r^k(r-1)^k=r^k\left[\left\{\binom{k}{0}r^k+\binom{k}{1}r^{k-1}+\binom{k}{2}r^{k-2}+\binom{k}{3}r^{k-3}+.......+\binom{k}{k}\right\}\right.$$
$$\left.-\left\{\binom{k}{0}r^k-\binom{k}{1}r^{k-1}+\binom{k}{2}r^{k-2}-\binom{k}{3}r^{k-3}+.......+(-1)^k\binom{k}{k}\right\}\right]$$

$$=r^k\left[\left\{\binom{k}{0}r^k+\binom{k}{1}r^{k-1}+\binom{k}{2}r^{k-2}+\binom{k}{3}r^{k-3}+....+\binom{k}{k}\right\}\right.$$
$$\left.-\left\{\binom{k}{0}r^k-\binom{k}{1}r^{k-1}+\binom{k}{2}r^{k-2}-\binom{k}{3}r^{k-3}+.......-\binom{k}{k}\right\}\right]$$
$$=r^k\left[2\left\{\binom{k}{1}r^{k-1}+\binom{k}{3}r^{k-3}+\binom{k}{5}r^{k-5}+.......+\binom{k}{k}\right\}\right]$$

$$\Rightarrow (r+1)^k r^k-r^k(r-1)^k=2\left\{\binom{k}{1}r^{2k-1}+\binom{k}{3}r^{2k-3}+\binom{k}{5}r^{2k-5}+.......+\binom{k}{k}r^k\right\} \quad \textbf{(32)}$$

Putting $r=1,2,3,.......,n$ in result **(32)**, we get

$$2^k.1^k-1^k.0^k=2\left\{\binom{k}{1}.1^{2k-1}+\binom{k}{3}.1^{2k-3}+\binom{k}{5}.1^{2k-5}+.......+\binom{k}{k}.1^k\right\} \quad \textbf{(32.1)}$$

$$3^k.2^k-2^k.1^k=2\left\{\binom{k}{1}.2^{2k-1}+\binom{k}{3}.2^{2k-3}+\binom{k}{5}.2^{2k-5}+.......+\binom{k}{k}.2^k\right\} \quad \textbf{(32.2)}$$

$$4^k.3^k - 3^k.2^k = 2\left\{\binom{k}{1}.3^{2k-1} + \binom{k}{3}.3^{2k-3} + \binom{k}{5}.3^{2k-5} + ... + \binom{k}{k}.3^k\right\} \qquad \textbf{(32.3)}$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$(n+1)^k n^k - n^k(n-1)^k = 2\left\{\binom{k}{1}n^{2k-1} + \binom{k}{3}n^{2k-3} + \binom{k}{5}n^{2k-5} + ... + \binom{k}{k}n^k\right\} \qquad \textbf{(32.}\boldsymbol{n}\textbf{)}$$

Adding **(32.1)**, **(32.2)**, **(32.3)**,……and **(32.*n*)**, we get

$$(n+1)^k n^k - 1^k.0^k = 2\left\{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + ... + \binom{k}{k}\sum_{r=1}^{n} r^k\right\}$$

$$\Rightarrow 2^k\left\{\frac{n(n+1)}{2}\right\}^k = 2\left\{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + ... + \binom{k}{k}\sum_{r=1}^{n} r^k\right\}$$

$$\Rightarrow 2^{k-1}\left\{\frac{n(n+1)}{2}\right\}^k = \left\{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + ... + \binom{k}{k}\sum_{r=1}^{n} r^k\right\}$$

$$\Rightarrow \left(\frac{n(n+1)}{2}\right)^k = \frac{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + ... + \binom{k}{k}\sum_{r=1}^{n} r^k}{2^{k-1}}$$

$$\Rightarrow \left(\sum_{r=1}^{n} r\right)^k = \frac{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + ... + \binom{k}{k}\sum_{r=1}^{n} r^{2k-k}}{\binom{k}{1} + \binom{k}{3} + \binom{k}{5} + ... + \binom{k}{k}} \qquad \textbf{(33)}$$

(ii) When k is even positive integer:

$$(r+1)^k r^k - r^k(r-1)^k = r^k\left[\left\{\binom{k}{0}r^k + \binom{k}{1}r^{k-1} + \binom{k}{2}r^{k-2} + \binom{k}{3}r^{k-3} + ... + \binom{k}{k}\right\}\right.$$

$$\left.-\left\{\binom{k}{0}r^k - \binom{k}{1}r^{k-1} + \binom{k}{2}r^{k-2} - \binom{k}{3}r^{k-3} + ... + (-1)^k\binom{k}{k}\right\}\right]$$

$$= r^k\left[\left\{\binom{k}{0}r^k + \binom{k}{1}r^{k-1} + \binom{k}{2}r^{k-2} + \binom{k}{3}r^{k-3} + ... + \binom{k}{k}\right\}\right.$$

$$\left.-\left\{\binom{k}{0}r^k - \binom{k}{1}r^{k-1} + \binom{k}{2}r^{k-2} - \binom{k}{3}r^{k-3} + ... + \binom{k}{k}\right\}\right]$$

$$= r^k\left[2\left\{\binom{k}{1}r^{k-1} + \binom{k}{3}r^{k-3} + \binom{k}{5}r^{k-5} + ... + \binom{k}{k-1}\right\}\right]$$

$$\Rightarrow (r+1)^k r^k - r^k (r-1)^k = 2\left\{\binom{k}{1} r^{2k-1} + \binom{k}{3} r^{2k-3} + \binom{k}{5} r^{2k-5} + \ldots + \binom{k}{k-1} r^k\right\} \qquad \textbf{(34)}$$

Putting $r = 1, 2, 3, \ldots\ldots, n$ in result **(34)**, we get

$$2^k.1^k - 1^k.0^k = 2\left\{\binom{k}{1}.1^{2k-1} + \binom{k}{3}.1^{2k-3} + \binom{k}{5}.1^{2k-5} + \ldots\ldots + \binom{k}{k-1}.1^k\right\} \qquad \textbf{(34.1)}$$

$$3^k.2^k - 2^k.1^k = 2\left\{\binom{k}{1}.2^{2k-1} + \binom{k}{3}.2^{2k-3} + \binom{k}{5}.2^{2k-5} + \ldots\ldots + \binom{k}{k-1}.2^k\right\} \qquad \textbf{(34.2)}$$

$$4^k.3^k - 3^k.2^k = 2\left\{\binom{k}{1}.3^{2k-1} + \binom{k}{3}.3^{2k-3} + \binom{k}{5}.3^{2k-5} + \ldots\ldots + \binom{k}{k-1}.3^k\right\} \qquad \textbf{(34.3)}$$

$$\ldots$$

$$\ldots$$

$$\ldots$$

$$(n+1)^k n^k - n^k (n-1)^k = 2\left\{\binom{k}{1} n^{2k-1} + \binom{k}{3} n^{2k-3} + \binom{k}{5} n^{2k-5} + \ldots\ldots + \binom{k}{k-1} n^k\right\} \qquad \textbf{(34.}\boldsymbol{n}\textbf{)}$$

Adding **(34.1)**, **(34.2)**, **(34.3)**,……and **(34.*n*)**, we get

$$(n+1)^k n^k - 1^k.0^k = 2\left\{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + \ldots\ldots + \binom{k}{k-1}\sum_{r=1}^{n} r^k\right\}$$

$$\Rightarrow (n+1)^k n^k = 2\left\{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + \ldots\ldots + \binom{k}{k-1}\sum_{r=1}^{n} r^k\right\}$$

$$\Rightarrow 2^k \left\{\frac{n(n+1)}{2}\right\}^k = 2\left\{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + \ldots\ldots + \binom{k}{k-1}\sum_{r=1}^{n} r^k\right\}$$

$$\Rightarrow 2^{k-1} \left\{\frac{n(n+1)}{2}\right\}^k = \left\{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + \ldots\ldots + \binom{k}{k-1}\sum_{r=1}^{n} r^k\right\}$$

$$\Rightarrow \left(\frac{n(n+1)}{2}\right)^k = \frac{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + \ldots\ldots + \binom{k}{k-1}\sum_{r=1}^{n} r^k}{2^{k-1}}$$

$$\Rightarrow \left(\sum_{r=1}^{n} r\right)^k = \frac{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + \ldots\ldots\ldots + \binom{k}{k-1}\sum_{r=1}^{n} r^{2k-k}}{\binom{k}{1} + \binom{k}{3} + \binom{k}{5} + \ldots\ldots + \binom{k}{k}} \qquad \textbf{(35)}$$

5. Conclusion

Results **(33)**and **(35)**are now combined and generalized as follows:

$$\frac{\binom{k}{1}\sum_{r=1}^{n} r^{2k-1} + \binom{k}{3}\sum_{r=1}^{n} r^{2k-3} + \binom{k}{5}\sum_{r=1}^{n} r^{2k-5} + \ldots\ldots\ldots}{\binom{k}{1} + \binom{k}{3} + \binom{k}{5} + \ldots\ldots\ldots} = \left(\sum_{r=1}^{n} r\right)^{k} \qquad \textbf{(36)}$$

So, the above result **(36)**is proved for all positive integral values of *k*.

Hence, the k^{th} power of the sum of the first *n* natural numbers is the Weighted Arithmetic Mean of the sum of their (2*k*-1)th powers, the sum of their (2*k*-3)th powers, the sum of their (2*k*-5)th powers,………with weights $\binom{k}{1}$, $\binom{k}{3}$, $\binom{k}{5}$,…...respectively ; where *k* is a positive integer.

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