Inverse Aboodh Transform of some standard functions and properties with Applications in ODE

Chetana Vilas Visave

Assistant Professor

Department of Mathematics,

Bhavan's College, Andheri(W), Mumbai.

Abstract : In this paper we study the definition of inverse Aboodh transform. We also study the inverse Aboodh transform of some standard functions .Further, we establish and prove some important properties related to inverse Aboodh transform. In addition, we establish application of inverse Aboodh transform to find particular solution of first and second order linear ordinary differential equations.

Keywords–Inverse Aboodh transform, Linearity, Change of scale, Effect of multiplication.

1. INTRODUCTION

Aboodh Transform was introduced by Khalid Suliman Aboodh in 2013 from classical Fourier integral. Aboodh transform is defined for function of exponential order by

$$A\{f(t)\} = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt,\ \ t \geq 0,\ \ k_1 \leq v \leq k_2,$$

where $f(t)$ is a function from the set of the form $A = \{f(t) : \exists\ M, k_1, k_2 > 0, |f(t)| < Me^{-vt}\}$.[1]

One of the most importance of transforms is solving ordinary differential equations, partial differenatial equations, linear volterra integro-differential equations etc. Aboodh transform is useful in solving ordinary differential equations, partial differenatial equations equations [2] which gives rise to make use of inverse Aboodh transform. Aboodh transform is also useful for solving solving fourth order parabolic PDE with variable coefficients [3]. The higher versions of Aboodh Transform such as double aboodh transform, triple aboodh transform are already established [8,9]. In addition to it several theorems and properties related to them are also verified. Moreover applications of these higher versions of aboodh transforms to solve integral, partial and fractional differential equations are also discussed[2,4,7,8,9,10].

In this article we introduce An Inverse Aboodh Transform in a different and more simpler way. We also prove some important properties for Inverse Aboodh Transform using some basic properties of Aboodh transform. We also illustrate use of Inverse Aboodh Transform in solving first and second order linear differential equations.

2. PRELIMINARIES

In this section, we study the definition of inverse aboodh transform.

Definition If $F(t)$ is piecewise continuous and of exponential order for t≥0 such that $A\{F(t)\} = f(v)$ then $F(t)$ is called inverse Aboodh transform of $f(v)$ and we write

$$A^{-1}\{f(v)\} = F(t)$$

3. INVERSE ABOODH TRANSFORM OF SOME STANDARD FUNCTIONS

In this section, we present the inverse Aboodh transform of some standard functions.

a) $A^{-1}\left\{\frac{1}{v^2}\right\} = 1,$

b) $A^{-1}\left\{\frac{1}{v^3}\right\} = t,$

In general, we can define

$$A^{-1}\left\{\frac{1}{v^{n+2}}\right\} = \frac{t^n}{n!}\ , n = 0, 1, 2, \dots$$

Explanation:-

As we know, [1] $A\left\{\frac{t^n}{n!}\right\} = \frac{1}{n!}\ A\{t^n\}$

$$= \frac{1}{n!}\frac{n!}{v^{n+2}}$$

$$= \frac{1}{v^{n+2}}$$

Thus, $A^{-1}\left\{\frac{1}{v^{n+2}}\right\} = \frac{t^n}{n!}\ , n = 0, 1, 2, \dots$

Alternatively, we can also prove it in following manner

$$[1] A\left\{\frac{t^n}{n!}\right\} = \frac{1}{v}\int_0^\infty \frac{t^n}{n!}e^{-vt}dt$$

$$= \frac{1}{n!}\left\{\frac{1}{v}\int_0^\infty t^n\ e^{-vt}dt\right\}$$

$$= \frac{1}{n!}\ A\{t^n\} = \frac{1}{v^{n+2}}.$$

Thus, $A^{-1}\left\{\frac{1}{v^{n+2}}\right\} = \frac{t^n}{n!}, n = 0, 1, 2, \dots$

c) $A^{-1}\left\{\frac{1}{v(v-a)}\right\} = A^{-1}\left\{\frac{1}{v^2-av}\right\} = e^{at}$

d) $A^{-1}\left\{\frac{1}{v(v^2+a^2)}\right\} = \frac{1}{a}\, sinat$

Explanation:

Consider,

$$[1]\ A\{\tfrac{1}{a}\, sinat\} = \frac{1}{a} A\{sinat\} = \frac{1}{a}\frac{a}{v(v^2+a^2)} = \frac{1}{v(v^2+a^2)}$$

$$\therefore A\left\{\frac{1}{a}\, sinat\right\} = \frac{1}{v(v^2+a^2)}$$

$$\Rightarrow A^{-1}\left\{\frac{1}{v(v^2+a^2)}\right\} = \frac{1}{a}\, sinat$$

e) $A^{-1}\left\{\frac{1}{(v^2+a^2)}\right\} = cosat$

f) $A^{-1}\left\{\frac{1}{v(v^2-a^2)}\right\} = \frac{1}{a}\, sinhat$

Explanation:

Consider,

$$[1]\ A\{\tfrac{1}{a}\, sinhat\} = \frac{1}{a} A\{sinhat\} = \frac{1}{a}\frac{a}{v(v^2-a^2)} = \frac{1}{v(v^2-a^2)}$$

$$\therefore A\left\{\frac{1}{a}\, sinat\right\} = \frac{1}{v(v^2-a^2)}$$

$$\Rightarrow A^{-1}\left\{\frac{1}{v(v^2-a^2)}\right\} = \frac{1}{a}\, sinhat$$

g) $A^{-1}\left\{\frac{1}{(v^2-a^2)}\right\} = coshat$

Thus we can summarize this in tabular form as

Function	Inverse Aboodh Transform
$\frac{1}{v^2}$	1
$\frac{1}{v^{n+2}}$	$\frac{t^n}{n!}, n = 0, 1, 2, \dots$
$\frac{1}{v(v-a)}$	e^{at}
$\frac{1}{v(v^2+a^2)}$	$\frac{1}{a}\, sinat$
$\frac{1}{(v^2+a^2)}$	$cosat$
$\frac{1}{v(v^2-a^2)}$	$\frac{1}{a}\, sinhat$
$\frac{1}{(v^2-a^2)}$	$coshat$

4. PROPERTIES OF INVERSE ABOODH TRANSFORM

In this section we discuss some properties of inverse Aboodh transform.

4.1 Linearity Property

If $f_1(v)$ and $f_2(v)$ are two functions such that $A^{-1}\{f_1(v)\}$ and $A^{-1}\{f_2(v)\}$ exists and c_1, c_2 are arbitrary constants then

$$A^{-1}\{c_1 f_1(v) + c_2 f_2(v)\} = c_1 A^{-1}\{f_1(v)\} + c_2 A^{-1}\{f_2(v)\}$$

Proof: Suppose, $A^{-1}\{f_1(v)\} = F_1(t)$

$A^{-1}\{f_2(v)\} = F_2(t)$

Then, $A\{F_1(t)\} = f_1(v)$ and $A\{F_2(t)\} = f_2(v)$
For constants c_1 *and* c_2,
[1]By linearity property of Aboodh transform, we have

$$A\{c_1F_1(t) + c_2F_2(t)\} = c_1A\{F_1(t)\} + c_2A\{F_2(t)\}$$
$$= c_1f_1(v) + c_2f_2(v)$$

Thus, $A^{-1}\{c_1f_1(v) + c_2f_2(v)\} = c_1F_1(t) + c_2F_2(t)$
$$= c_1A^{-1}\{f_1(v)\} + c_2A^{-1}\{f_2(v)\}.$$

Generalization:

The Linearity property can be generalized for n-functions $f_i(v), i = 1, 2, \dots, n$ whose inverse Aboodh transform exists as follows

$$A^{-1}\{\textstyle\sum_{i=1}^{n} c_i f_i(v)\} = \sum_{i=1}^{n} c_i A^{-1}\{f_i(v)\}$$

Remark:

If we substitute $c_2 = 0$ in Linearity property,

$$A^{-1}\{c_1f_1(v) + c_2f_2(v)\} = c_1A^{-1}\{f_1(v)\} + c_2A^{-1}\{f_2(v)\}$$

We get,

$$A^{-1}\{c_1f_1(v)\} = c_1A^{-1}\{f_1(v)\}$$

Thus, we can conclude that "Any constant multiplier can be taken out while finding an inverse Aboodh transform".

4.2 Change of scale property

If $A^{-1}\{f(v)\} = F(t)$ then $A^{-1}\{f(av)\} = \frac{1}{a^2}F(\frac{t}{a})$

Proof: As $A^{-1}\{f(v)\} = F(t)$,
$$\Rightarrow f(v) = A\{F(t)\}$$
$$\therefore f(v) = \frac{1}{v}\int_0^\infty F(t)e^{-vt}dt$$

Thus,
$$f(av) = \frac{1}{av}\int_0^\infty F(t)e^{-avt}dt$$
$$= \frac{1}{av}\int_0^\infty F(t)e^{-(at)v}dt$$

Put, $at = u \Rightarrow a\,dt = du$
$$\Rightarrow dt = \frac{du}{a}$$
$$\therefore f(av) = \frac{1}{av}\int_0^\infty F(t)e^{-avt}dt = \frac{1}{av}\int_0^\infty e^{-uv}F\left(\frac{u}{a}\right)\frac{du}{a}$$
$$= \frac{1}{a^2}\frac{1}{v}\int_0^\infty e^{-uv}F\left(\frac{u}{a}\right)du$$

Replacing u by t, we have
$$= \frac{1}{a^2}\frac{1}{v}\int_0^\infty e^{-vt}F\left(\frac{t}{a}\right)dt$$

Thus, [1] $\quad f(av) = \frac{1}{a^2}A\{F(\frac{t}{a})\}$

Applying inverse Aboodh transform on both sides,
$$A^{-1}\{f(av)\} = A^{-1}\left\{\frac{1}{a^2}A\left\{F\left(\frac{t}{a}\right)\right\}\right\}$$
$$= \frac{1}{a^2}A^{-1}[A\{F(\frac{t}{a})\}]$$
$$A^{-1}\{f(av)\} = \frac{1}{a^2}F(\frac{t}{a})$$

4.3 Effect of multiplication by e^{-av}

If $A^{-1}\{f(v)\} = F(t)$ then, $A^{-1}\{e^{-av}f(v)\} = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

Proof: Consider, $A^{-1}\{f(v)\} = F(t)$
$$[1] \Rightarrow f(v) = A\{F(t)\}$$
$$= \frac{1}{v}\int_0^\infty e^{-vt}F(t)dt$$
$$\therefore f(v) = \frac{1}{v}\int_0^\infty e^{-vt}F(t)dt$$
$$\Rightarrow e^{-av}f(v) = e^{-av}\frac{1}{v}\int_0^\infty e^{-vt}F(t)dt$$
$$\Rightarrow e^{-av}f(v) = \frac{1}{v}\int_0^\infty e^{-(a+t)v}F(t)dt$$

Substitute, $t + a = u \Rightarrow dt = du$

Thus,
$$e^{-av}f(v) = \frac{1}{v}\int_a^\infty e^{-uv}F(u-a)du$$
$$= \frac{1}{v}\int_0^a e^{-uv}0du + \frac{1}{v}\int_a^\infty e^{-uv}F(u-a)du$$

Replace u by t, gives
$$e^{-av}f(v) = \frac{1}{v}\int_0^a e^{-vt}0dt + \frac{1}{v}\int_a^\infty e^{-vt}F(t-a)dt$$
$$= \frac{1}{v}\int_0^\infty G(t)e^{-vt}dt$$

Where, $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

Thus, $\quad e^{-av}f(v) = A\{G(t)\}$

$$\Rightarrow A^{-1}\{e^{-av}f(v)\} = G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

4.4 Effect of multiplication by v

If $A^{-1}\{f(v)\} = F(t)$ *and* $F(0) = 0$ then $A^{-1}\{vf(v)\} = F'(t)$.

In other words, the effect of multiplication by v to $f(v)$ is equivalent to differentiation of $F(t)$ provided $F(0) = 0$.

Proof: Let, $A^{-1}\{f(v)\} = F(t), F(0) = 0$

$$\Rightarrow A\{F(t)\} = f(v).$$

We know,[1] $A\{F'(t)\} = vA\{F(t)\} - \frac{F(0)}{v}$

$$= vA\{F(t)\} - 0 \ \{\because F(0) = 0\}$$

$$\therefore A\{F'(t)\} = vf(v)$$

Hence,

$$A^{-1}\{vf(v)\} = F'(t)$$

4.5 Effect of multiplication by v^n

The repeated multiplication by v to $f(v)$ together with the assumption that $F(0) = F'(0) = \cdots = F^{(n-1)}(0) = 0$ leads to the result

$$A^{-1}\{v^n f(v)\} = F^{(n)}(t)$$

Proof: We know,

$$[1]\ A\{F^{(n)}(t)\} = v^n A\{F(t)\} - \sum_{k=0}^{n-1} \frac{F^{(k)}(0)}{v^{2-n+k}}$$

$$\Rightarrow A\{F^{(n)}(t)\} = v^n A\{F(t)\} \text{ with assumption that } F(0) = F'(0) = \cdots = F^{(n-1)}(0) = 0$$

$$= v^n f(v)$$

Thus,

$$A^{-1}\{v^n f(v)\} = F^{(n)}(t) \text{ provided } F(0) = F'(0) = \cdots = F^{(n-1)}(0) = 0$$

5 Examples

1. $A^{-1}\{\frac{e^{-\pi v}}{v^2+9}\}$

Solution: Let, $f(v) = \frac{1}{v^2+9}$

Then,

$$A^{-1}\{f(v)\} = A^{-1}\{\frac{1}{v^2+9}\}$$

$$= cos3t = F(t)$$

$$\therefore A^{-1}\left\{e^{-\pi v}\frac{1}{v^2+9}\right\} = A^{-1}\{e^{-\pi v}f(v)\}$$

$$= G(t)$$

$$= \begin{cases} F(t-\pi), & t > \pi \\ 0, & t < \pi \end{cases}$$

$$A^{-1}\{\frac{e^{-\pi v}}{v^2+9}\} = \begin{cases} cos3(t-\pi), & t > \pi \\ 0, & t < \pi \end{cases}$$

2. $A^{-1}\left\{\frac{1}{(av)^{n+2}}\right\}$

Solution: We know, $A^{-1}\left\{\frac{1}{v^{n+2}}\right\} = \frac{t^n}{n!}, n = 0, 1, 2, \ldots$

Then, $A^{-1}\left\{\frac{1}{(av)^{n+2}}\right\} = \frac{1}{a^2}\frac{(\frac{t}{a})^n}{n!}$

$$= \frac{1}{a^{n+2}}\frac{t^n}{n!}, \ n = 0, 1, 2, \ldots,$$

3. $A^{-1}\left\{v^k \frac{1}{v^{n+2}}\right\}, k \le n$

Solution: Consider,

$$f(v) = \frac{1}{v^{n+2}}$$

$$A^{-1}\{f(v)\} = \frac{t^n}{n!}, n = 0, 1, 2, \ldots$$

Clearly, $F(t) = \frac{t^n}{n!}$, has $F(0) = F'(0) = \cdots = F^{(n-1)}(0) = 0$

Hence, $A^{-1}\{vf(v)\} = F'(t)$

$A^{-1}\{v^2 f(v)\} = F''(t)$ and so on

$A^{-1}\{v^k f(v)\} = F^k(t)$

Therefore,

$$A^{-1}\left\{v^k \frac{1}{v^{n+2}}\right\} = F^k(t), k \le n.$$

6 . Applications of Inverse Aboodh transform

Inverse Aboodh transform is useful in obtaining the particular solutions of first and second order linear ordinary differential equations.

6.1 Consider Initial value problem

$$\frac{d^2y}{dt^2} + y = 0, y(0) = 1, y'(0) = 0$$

To obtain the solution of this O.D.E.

Apply Aboodh transform on both sides, gives

$$A\{y''(t)\} + A\{y(t)\} = 0$$

[1] $\Rightarrow v^2 A\{y(t)\} - \frac{y'(0)}{v} - y(0) + A\{y(t)\} = 0$

$$\Rightarrow (v^2 + 1)A\{y(t)\} - 1 = 0$$

$$\Rightarrow A\{y(t)\} = \frac{1}{v^2+1}$$

Applying inverse Aboodh transform on both sides, gives

$$y(t) = A^{-1}\left\{\frac{1}{v^2+1}\right\} = cost$$

Thus the solution to I.V.P. is $y(t) = cost$.

6.2 Consider Initial value problem

$$\frac{d^2y}{dt^2} - y = 6, y(0) = 1, y'(0) = 0$$

To obtain the solution of this O.D.E.

Apply Aboodh transform on both sides, gives

[1] $A\{y''(t)\} - A\{y(t)\} = 6$

$$\Rightarrow v^2 A\{y(t)\} - \frac{y'(0)}{v} - y(0) + A\{y(t)\} = 6$$

$$\Rightarrow (v^2 - 1)A\{y(t)\} - 1 = 6$$

$$\Rightarrow (v^2 - 1)A\{y(t)\} = 7$$

$$\Rightarrow A\{y(t)\} = \frac{7}{v^2-1}$$

Applying inverse Aboodh transform on both sides, gives

$$y(t) = 7A^{-1}\left\{\frac{1}{v^2-1}\right\} = 7cosht$$

REFERENCES

[1] Khalid Suliman Aboodh, 2013. The New Integral Transform "Aboodh Transform", Global Journal of Pure and Applied Mathematics. Volume 9, Number 1 (2013), pp. 35-43.

[2] Khalid Suliman Aboodh, Application of New Transform "Aboodh Transform" to Partial Differentail Equations, Global Journal of Pure and Applied Mathematics,ISSN 0973-1768 Volume10, Number 2(2014), pp.249-254.

[3] K.S.Aboodh, Solving fourth order PDE with variable coefficients using Aboodh transform homotopy perturbation method, Pure and Applied Mathematics Journal, 4(5) (2015) 219-224

[4] K. S. Aboodh, , R. A. Farah, I. A. Almardy A. K. Osman, SOLVING DELAY DIFFERENTIAL EQUATIONS BY ABOODH TRANSFORMATION METHOD, International Journal of Applied Mathematics Statistical Sciences (IJAMSS) ISSN(P): 2319-3972; ISSN(E): 2319-3980 Vol. 7, Issue 2, Feb - Mar 2018; 55 – 64

[5] Khalid Suliman Aboodh. Solving Porous Medium Equation Using Aboodh Transform Homotopy Perturbation Method. American Journal of Applied Mathematics. Vol. 4, No. 6, 2016, pp. 271-276. doi: 10.11648/j.ajam.20160406.1

[6] K. S. Aboodh, R. A. Farah, I. A. Almardy and F. A. Almostafa, Solution of Telegraph Equation by Using Double Aboodh Transform, Elixir International Journal ISSN:2229-712x 110(2017) 48213-48217.

[7] Mohand, M, Khalid Suliman Aboodh, Abdelbagy, A. On the Solution of Ordinary Differential Equation with Variable Coefficients using Aboodh Transform, Advances in Theoretical and Applied Mathematics ISSN 0973-4554 Volume 11, Number 4 (2016), pp. 383-389.

[8] Nuruddeen R. I. and Nass A. M., Aboodh decomposition method and its application in solving linear and nonlinear heat equations, European Journal of Advances in Engineering and Technology, 3(7) (2016) 34-37

[9] S. Alfaqeih, T.ÖZİŞ Note on First Aboodh Transform of Fractional order and Its Properties, International Journal of Progressive Sciences and Technologies (IJPSAT) ISSN: 2509-0119. Volume 13 No. 2 March 2019, pp. 252-256

[10] S. Alfaqeih, T.ÖZİŞ Note on Double Aboodh Transform of Fractional order and Its Properties, OMJ, 01 (01): 114, ISSN:2672-7501

[11] S. Alfaqeih, T.ÖZİŞ Note on Triple Aboodh Transform and Its Application,International Journal of Engineering and Information Systems (IJEAIS) ISSN: 2000-000X Vol. 3 Issue 3, March 2019, Pages: 41-50