A NOTE ON ANALYTIC FUNCTIONS AND A NEW DIFFERENTIAL OPERATOR

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Abstract:

In this paper, we study the new linear differential operator to the analytic functions. Also, we discuss the results about the analytic functions using the new differential operator.

Keywords: Differential equation, Complex Analysis and Analytic theory.

1. INTRODUCTION

The main goal of this topic is to define and give some of the important properties of complex analytic functions. A function $f(z)$ is analytic if it has a complex derivative $f'(z)$. In general, the rules for computing derivatives will be familiar to you from single variable calculus. However, a much richer set of conclusions can be drawn about a complex analytic function than is generally true about real differentiable functions.

1.1 The derivative: preliminaries

In calculus we defined the derivative as a limit. In complex analysis we will do the same.

$$f'(z) = \lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z} = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

Before giving the derivative our full attention we are going to have to spend some time exploring and understanding limits. To motivate this well first look at two simple examples one positive and one negative.

Example : 1.1

Find the derivative of $f(z) = z^2$.

answer:

We compute using the definition of the derivative as a limit.

$$\lim_{\Delta z\to 0}\frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0}\frac{z^2+2z\Delta z+(\Delta z)^2 - z^2}{\Delta z}$$
$$= \lim_{\Delta z\to 0} 2z+\Delta z = 2z$$

That was a positive example. Here's a negative one which shows that we need to a careful understanding of limits.

Example: 1.2

Let $f(z) = \bar{z}$. Show that the limit for $f'(0)$ does not converge.

Answer:

Let's try to compute $f'(0)$ using a limit:

$$f'(0) = \lim_{\Delta z\to 0}\frac{f(\Delta z)-f(0)}{\Delta z} = \lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Here we used $\Delta z = \Delta x + \Delta y$

Now, $\Delta z \to 0$ means both Δx and Δy have to go to 0. There are lots of ways to do this. For example, if we let Δz go to 0 along the x-axis then, $\Delta y = 0$ while Δx goes to 0. In this case, we would have$f'(0) = \lim_{\Delta x\to 0}\frac{\Delta x}{\Delta y} = 1.$ On the other hand, if we let Δz go to 0 along the positive y-axis then

$$f'(0) = \lim_{\Delta y\to 0}\frac{-i\Delta y}{i\Delta y} = -1$$

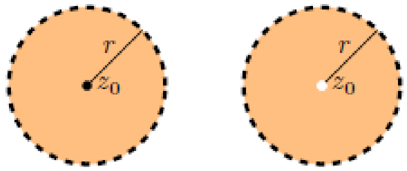
The limit don't agree! The problem is that the limit depends on how Δz apporaches 0. If we came from other directions we'd get other values. There's nothing to do, but agree that the limit does not exist.

We'll there is something we can do: explore and understand limits. Let's do that now.

1.2 Open disks, open deleted disks, open regions

Definition: 1.3

The open disk of radius r around z_0 is the set of points z with $|z - z_0| < r$, i.e., all points within distance r of z_0. The open deleted disk of radius r around z_0 is the set of points z with $0 < |z - z_0| < r$. That is, we remove the center z_0 from the open disk.



Left: an open disk around z_0; right: a deleted open disk around z_0

A deleted disk is also called a punctured disk.

Definition: 1.4

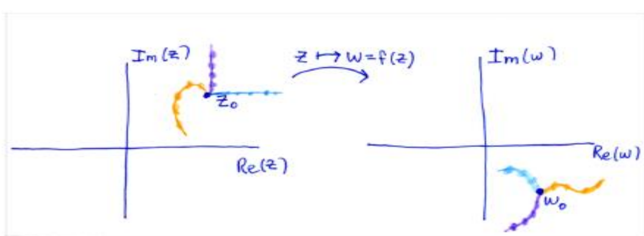
An open region in the complex plane is a set A with the property that every point in A can be surrounded by an open disk that lies entirely in A. We will often drop the word open and simply call A a region. In the figure below, the set A on the left is an open region because for every point in A we can draw a little circle around the point that is completely in A. (The dashed boundary line indicates that the boundary of A is not part of A). In contrast, the set B is not an open region. Notice the point z shown is on the boundary, so every disk around z contains points outside B.

Left: an open region A; right: B is not an open region

Definition: 1.4

If $f(z)$ is defined on a deleted disk around z_0 then we say $\lim_{z \to z_0} f(z) = w_0$

if $f(z)$ goes to w_0 no matter what direction z approaches z_0. The figure below shows several sequences of points that approach z_0. If $\lim_{z \to z_0} f(z) = w_0$ then $f(z)$ must go to w_0 along each of these sequences.



Sequences going to z_0 are mapped to sequences going to w_0.

Definition: 1.6

Many functions have obvious limits. For example

$$\lim_{z \to 2} z^2 = 4 \text{ and } \lim_{z \to 2} (z^2 + 2)/(z^3 + 1) = 6/9$$

Here is an example where the limit doesn't exist because different sequences give different limits.

Example: 1.7

(No limit) Show that $\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{z \to 0} \frac{x+iy}{x-iy}$ doest not exist

Answer:

On the real axis we have $\frac{z}{\bar{z}} = \frac{x}{x} = 1$, so the limit as $z \to 0$ along the real axis is 1.

On the imaginary axis we have $\frac{z}{\bar{z}} = \frac{iy}{-iy} = -1$, so the limit as $z \to 0$ along the imaginary axis is -1. Since the two limits do not agree the limit as $z \to 0$ does not exist.

1.5 Properties of limits

We have the usual properties of limits. Suppose

$$\lim_{z \to z_0} f(z) = w_1 \text{ and } \lim_{z \to z_0} g(z) = w_2$$

Then

- $\lim_{z \to z_0} f(z) + g(z) = w_1 + w_2$
- $\lim_{z \to z_0} f(z)\, g(z) = w_1 . w_2$
- If $w_2 \neq 0$ then $\lim_{z \to z_0} f(z) \backslash g(z) = w_1 \backslash w_2$
- If $h(z)$ is continuous and defined on a neighborhood of w_1 then $\lim_{z \to z_0} h\big(f(z)\big) = h(w_1)$

  (Note: we will give the official definition of continuity in the next section)

We won't give a proof of these properties. As a challenge, you can try to supply it using the formal definition of limits given in the appendix.

We can restate the definition of limit in terms of functions of (x, y)

Write $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Then

$$\lim_{z \to z_0} f(z) = w_0 \text{ iff } \begin{cases} \lim_{P \to P_0} u(x, y) = u_0 \\ \lim_{P \to P_0} v(x, y) = v_0 \end{cases}$$

Where $P = (x, y), P_0 = (x_0, y_0), w_0 = u_0 + iv_0$.

1.6 Continuous functions

Definition: 1.8

If the function $f(z)$ is defined on an open disk around z_0 and $\lim_{z \to z_0} f(z) = f(z_0)$ then we say f is continuous at z_0.

If f is defined on an open region A then the phrase 'f is continuous on A' means that f is continuous at every point in A.

As usual, we can rephrase this in terms of functions of (x, y).

Example: 1.9

(i) A polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is continuous on the entire plane. Reason: it is clear that each power $(x + iy)^k$ is continuous as a function of (x, y).

(ii) The exponential function is continuous on the entire plane. Reason: $e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$. So the both the real and imaginary parts are clearly continuous as a function of (x, y).

1.7 Properties of continuous functions

Since continuity is defined in terms of limits, we have the following properties of continuous functions.

Suppose $f(z)$ and $g(z)$ are continuous on a region A. Then

- $f(z) + g(z)$ is continuous on A.
- $f(z)g(z)$ is continuous on A.
- $f(z)\backslash g(z)$ is continuous on A except (possibly) at points where $g(z) = 0$.
- If h is continuous on $f(A)$ then $h\big(f(x)\big)$ is continuous on A.

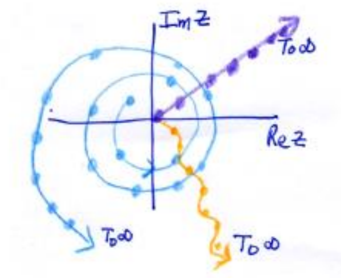
Using these properties we can claim continuity for each of the following functions:

- e^{x^2}
- $\cos(z) = \left(e^{iz} + e^{-iz}\right)/2$
- If $P(z)$ and $Q(z)$ are polynomials then $P(z)/Q(z)$ is continuous except at roots of $Q(z)$.

1.8 The point at infinity

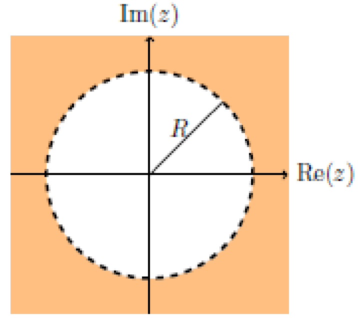
By definition the extended complex plane $= C \cup \{\infty\}$. That is, we have one point at infinity to be thought of in a submitting sense described as follows.

A sequence of points $\{x_n\}$ goes to infinity if $|z_n|$ goes to infinity. This "point a infinity" is approached in any direction we go. All of the sequences shown in the figure below are growing, so they all go to the (same) "point at infinity".



Various sequences all going to infinity

If we draw a large circle around 0 in the plane, then we call the region outside this circle a neighborhood of infinity.



The shaded region outside of radius R is a neighborhood of infinity.

2. CAUCHY RIEMANN EQUATION

The Cauchy-Riemann equations are our first consequences of the fact that the limit defining $f(z)$ must be the same no matter which direction you approach z from. The Cauchy-Riemann equations will be one of the most important tools in our toolbox.

2.1 Partial derivatives as limits

Before getting to the Cauchy-Riemann equations we remind you about partial derivatives. If $u(x, y)$ is a function of two variables then the partial derivatives of u are defined as

$$\frac{\partial u}{\partial x}(x, y) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$$

i.e., the derivative of u holding y constant.

$$\frac{\partial u}{\partial y}(x, y) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}$$

i.e., the derivative of u holding x constant.

2.2 The Cauchy-Riemann equations

The Cauchy-Riemann equations use the partial derivatives of u and v to do two things: first, check if f has a complex derivative and second, how to compute that derivative.

We start by stating the equations as a theorem.

Theorem: 2.1

(Cauchy-Riemann equations) If $f(z) = u(x, y) + iv(x, y)$ is differentiable then

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

In particular, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

This last set of partial differential equations is what is usually meant by the Cauchy-Riemann equations.

Here is the short form of the Cauchy-Riemann equations: $u_x = v_y$, $u_y = -v_x$

Proof:

Let's suppose that $f(z)$ is differentiable in some region A and

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

We'll compute $f'(z)$ by approaching z first from the horizontal direction and then from the vertical direction. We'll use the formula

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Where $\Delta z = \Delta x + i\Delta y$. Horizontal direction : $\Delta y = 0, \Delta z = \Delta x$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$
$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x+iy) - f(x+iy)}{\Delta x} \quad = \lim_{\Delta x \to 0} \frac{(u(x+\Delta x,y)+iv(x+\Delta x,y)) - (u(x,y)+iv(x,y))}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x,y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x,y) - v(x,y)}{\Delta x} \quad = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)$$

Vertical direction: $\Delta x = 0, \Delta z = i\Delta y$ (We'll do this one a little faster)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad = \lim_{\Delta y \to 0} \frac{(u(x,y+\Delta y)+iv(x,y+\Delta y)) - (u(x,y)+iv(x,y))}{i\Delta y} \quad = \lim_{\Delta y \to 0} \frac{u(x,y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x,y-\Delta y) - v(x,y)}{i\Delta y}$$
$$= \frac{1}{i}\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y) \qquad = \frac{\partial u}{\partial y}(x,y) - i\frac{\partial v}{\partial y}(x,y)$$

We have found two different represented of $f'(z)$ in terms of the partials of u and v.if put them together we have the Cauchy-Riemann equations:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \Longrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \text{and} - \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \text{QED}$$

It turns out that the converse is true and will be very useful to us.

Theorem: 2.2

Consider the function $f(z) = u(x,y) + iv(x,y)$ defined on a region A. If u and v satisfy the Cauchy-Riemann equations and have continuous partials then $f(z)$ is differentiable on A.The proof of this is a tricky exercise in analysis. It is somewhat beyond the scope of this class, so we will skip it. With a little effort you should be able to grasp it.

Theorem: 2.3

If $f(z)$ is differentiable on a disk and $f'(z) = 0$ on the disk then $f(z)$ is constant.

Proof:

Since f is differentiable and $f'(z) \equiv 0$, the Cauchy-Riemann equations show that

$$u_x(x,y) = u_y(x,y) = v_x(x,y) = v_y(x,y) = 0$$

We know from multivariable calculus that a function of (x,y) with both partials identically zero is constant. Thus u and v are constant, and therefore so is f.

3. ANALYTIC FUNCTION OF COMPLEX ORDER DEFINED BY NEW DIFFERENTIAL OPERATOR

Let H be the class of function analytic in $U\colon\{z\colon|z| < 1\}$ and let $H[a,n]$ be the subclasses of H consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$. Let A be the subclasses of H consisting of functions of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ or

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

Let $A(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n}^{\infty} a_{k+1} z^{k+1} \tag{2}$$

$$a_{k+1} \geq 0, n \in \{1,2,3, \dots \dots \dots \dots \dots .\}$$

Which are analytic in the open unit disk $U = \{z\colon|z| < 1\}$.

Next, we define (n,δ)-neighbourhood for the functions belonging to class $A(n)$ and also for identity function.

Definition: 3.1

((n,δ)-neighbourhood). By following the earlier inverstigations by Goodman and Ruscheweyh, for and $f(z) \in A(n)$ and $\delta \geq 0$, we define the (n,δ)-neighbourhood of f by

$$N_{n,\delta}(f) = \left\{g \in A(n)\colon g(z) = z - \sum_{k=n}^{\infty} b_{k+1} z^{k+1} \; and \; \sum_{k=n}^{\infty} (k+1)|a_{k+1} - b_{k+1}| \leq \delta\right\} \tag{3}$$

In particular for the identity function $e(z) = z$ we have

$$N_{n,\delta}(e) = \left\{ g \in A(n): g(z) = z - \sum_{k=n}^{\infty} b_{k+1} z^{k+1} \; and \; \sum_{k=n}^{\infty} (k+1)|b_{k+1}| \leq \delta \right\} \tag{4}$$

We say that the function $f(z) \in A(n)$ is said to be starlike functions of complex order γ or $f(z) \in S_n^*(\gamma)$ if it satisfies the inequality

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{z\big(f'(z)\big)}{f(z)} - 1\right)\right) > 0, z \in U, \gamma \in C\backslash\{0\} \tag{5}$$

Furthermore, a function $f(z) \in A(n)$ is said to be convex functions of complex order γ on $f(Z) \in C_n^*(\gamma)$ it is satisfies the inequality

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{z\big(f''(z)\big)}{f'(z)}\right)\right) > 0, z \in U, \gamma \in C\backslash\{0\} \tag{6}$$

The class $S_n^*(\gamma)$ and $C_n^*(\gamma)$ are essentially from the classes of starlike and convex functions of complex order, which were considered by Nasr and Aouf and Wiatrowsky respectively. Let $S_n(\gamma, \lambda, \beta)$ denote the subclass of $A(n)$ consisting of functions $f(z)$ which satisfy the following inequality:

$$\left|\frac{1}{\gamma}\left(\frac{\lambda z^3 f'''(z) + (1+2\lambda) z^2 f''(z) + z f'(z)}{\lambda z^2 f''(z) + z f'(z)} - 1\right)\right| < \beta$$

where $z \in U, \gamma \in C\backslash\{0\}, 0 \leq \lambda \leq 1, 0 < \beta \leq 1$.

Let $R_n(\gamma, \lambda, \beta)$ denote the subclass of $A(n)$ consisting of functions $f(z)$ which satisfy the following inequality

$$\left|\frac{1}{\gamma}\left(\lambda z^2 f'''(z) + (1+2\lambda) z f''(z) + f'(z) - 1\right)\right| < \beta$$

where $z \in U, \gamma \in C\backslash\{0\}, 0 \leq \lambda \leq 1, 0 < \beta \leq 1$.

The class $S_n(\gamma, \lambda, \beta)$ was studied by Kamali and Akbulut. Since A is the class of functions $f(z)$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the open unit disk $U = \{z: |z| < 1\}$.

For a function f in A we define the following differential operator:

$$\begin{aligned} D^0_{\lambda,v,\varrho}(\alpha,\omega) f(z) &= f(z), \\ D^1_{\lambda,v,\varrho}(\alpha,\omega) f(z) &= \left(\frac{v - (\varrho+\lambda)\omega^\alpha}{v}\right) f(z) + \left(\frac{(\varrho+\lambda)\omega^\alpha}{v}\right) z f'(z), \\ D^2_{\lambda,v,\varrho}(\alpha,\omega) f(z) &= D\left(D^1_{\lambda,v,\varrho}(\alpha,\omega) f(z)\right), \\ D^m_{\lambda,v,\varrho}(\alpha,\omega) f(z) &= D\left(D^{m-1}_{\lambda,v,\varrho}(\alpha,\omega) f(z)\right). \end{aligned} \tag{7}$$

If f given by (1), then from (7) we define the following differential operator

$$D^m_{\lambda,v,\varrho}(\alpha,\omega) f(z) = z + \sum_{k=2}^{\infty} \left(\frac{v + (k-1)(\varrho+\lambda)\omega^\alpha}{v}\right)^m a_k z^k \tag{8}$$

Where $f(z) \in A, v > 0, \varrho, \omega, \lambda, \alpha \geq 0, m \in N_0$

This operator generalizes certain differential operators such as:

(1) $v = 1, \varrho = 0$ we get

$$D^m_{\lambda,1,0}(\alpha,\omega) f(z) = z + \sum_{k=2}^{\infty} (1 + (k-1)\lambda w^\alpha)^m a_k z^k$$

of Darus and Faisal

(2) $\alpha = \omega = v = 1, \varrho = 0$ we get

$$D^m_{\lambda,1,0}(1,1) f(z) = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))^m a_k z^k$$

of A1-Oboudi

(3) $\alpha = \omega = v = \lambda = 1, \varrho = 0$ we get

$$D_{1,1,0}^{m}(1,1)f(z) = z + \sum_{k=2}^{\infty} (k)^m a_k z^k$$

of Salagean.

(4) $\alpha = \omega = v = 1, \lambda = 2, \varrho = 0$, we get

$$D_{2,1,0}^{m}(1,1)f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+1}{2}\right)^m a_k z^k$$

of Uralegaddi and Somanatha.

By using the same process, we can write the following equalities for the function $f(z)$ belonging to the class $A(z)$,

$$D_{\lambda,v,\varrho}^{0}(\alpha,\omega)f(z) = f(z)$$

$$D_{\lambda,v,\varrho}^{1}(\alpha,\omega)f(z) = \left(\frac{v-(\varrho+\lambda)\omega^\alpha}{v}\right)f(z) + \left(\frac{(\varrho+\lambda)\omega^\alpha}{v}\right)zf'(z), \tag{9}$$

$$D_{\lambda,v,\varrho}^{2}(\alpha,\omega)f(z) = D\left(D_{\lambda,v,\varrho}^{1}(\alpha,\omega)f(z)\right),$$

$$D_{\lambda,v,\varrho}^{\mho}(\alpha,\omega)f(z) = D\left(D_{\lambda,v,\varrho}^{\mho-1}(\alpha,\omega)f(z)\right).$$

If f given by (2), then from (9) we define the following differential operator

$$D_{\lambda,v,\varrho}^{\mho}(\alpha,\omega)f(z) = z - \sum_{k=n}^{\infty} \left(\frac{v+k(\varrho+\lambda)\omega^\alpha}{v}\right)^{\mho} a_{k+1} z^{k+1} \tag{10}$$

where $f \in A(n), v > 0, \varrho, \omega, \lambda, \alpha \geq 0, \mho \in N_0$

Finally, in the term of the generalized Salagean differential operator, let $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)$ denote the subclass of $A(n)$ consisting of the functions $f(z)$ which satisfy the inequality

$$\left|\frac{1}{\gamma}\left(\frac{(\mu)z\left(D_{\lambda,v,\varrho}^{\mho+3}(\alpha,\omega)f(z)\right)' + (1-\mu)z\left(D_{\lambda,v,\varrho}^{\mho+2}(\alpha,\omega)f(z)\right)'}{(\mu)z\left(D_{\lambda,v,\varrho}^{\mho+2}(\alpha,\omega)f(z)\right)' + (1-\mu)z\left(D_{\lambda,v,\varrho}^{\mho+1}(\alpha,\omega)f(z)\right)'} - 1\right)\right| < \beta \tag{11}$$

where $f \in A(n), \gamma \in C\backslash\{0\}, v > 0, \varrho, \omega, \lambda, \alpha, \mu \geq 0, \mho \in N_0, z \in U$. Also, let $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)$ denote the subclass of $A(n)$ consisting of the functions $f(z)$ which satisfy the inequality

$$\left|\frac{1}{\gamma}\left(\mu z\left(D_{\lambda,v,\varrho}^{\mho+3}(\alpha,\omega)f(z)\right)' + (1-\mu)z\left(D_{\lambda,v,\varrho}^{\mho+2}(\alpha,\omega)f(z)\right)' - 1\right)\right| < \beta \tag{12}$$

where $f \in A(n), \gamma \in C\backslash\{0\}, v > 0, \varrho, \omega, \lambda, \alpha, \mu \geq 0, \mho \in N_0, z \in U$.

Our main work here is to investigate the (n,δ)- neighborhood of the above said classes i.e. $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)$ and $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)$. Similar work has been seen for different subclasses done by other authors and of course many others.

3.2 Inclusion relations involving (n,δ)- neighborhood

Lemma: 3.2

Let the function $f(z) \in A(n)$ be defined by (2), then $f(z)$ is in the class $S_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho,\omega)$ if and only if

$$\sum_{k=n}^{\infty} \left(\frac{v+k(\varrho+\lambda)\omega^\alpha}{v}\right)^{\mho+1} \left(\frac{v+\mu k(\varrho+\lambda)\omega^\alpha}{v}\right) \times (k+1)\left(\frac{\beta|\gamma|v+k(\varrho+\lambda)\omega^\alpha}{v}\right) a_{k+1} \leq \beta|\gamma|, \tag{13}$$

where $f \in A(n), v > 0, \varrho, \omega, \lambda, \alpha \geq 0, \mho \in N_0, z \in U$.

Proof:

Let $f(z) \in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)$, then from (11) we have

$$\left|\frac{1}{\gamma}\left(\frac{(\mu)z\left(D_{\lambda}^{\mho+3}(v,\alpha,\omega)f(z)\right)' + (1-\mu)z\left(D_{\lambda}^{\mho+2}(v,\alpha,\omega)f(z)\right)'}{(\mu)z\left(D_{\lambda}^{\mho+2}(v,\alpha,\omega)f(z)\right)' + (1-\mu)z\left(D_{\lambda}^{\mho+1}(v,\alpha,\omega)f(z)\right)'} - 1\right)\right| < \beta,$$

where $f \in A(n), v > 0, \varrho, \omega, \lambda, \alpha \geq 0, \mho \in N_0$ or

$$\Re\left(\frac{(\mu)z\left(D_{\lambda}^{\mho+3}(v,\alpha,\omega)f(z)\right)' + (1-\mu)z\left(D_{\lambda}^{\mho+2}(v,\alpha,\omega)f(z)\right)'}{(\mu)z\left(D_{\lambda}^{\mho+2}(v,\alpha,\omega)f(z)\right)' + (1-\mu)z\left(D_{\lambda}^{\mho+1}(v,\alpha,\omega)f(z)\right)'} - 1\right) > -\beta|\gamma|,$$

where $f \in A(n), \gamma \in C\backslash\{0\}, v > 0, \varrho, \omega, \lambda, \alpha \geq 0, \mho \in N_0, z \in U$, after taking the limit when$z \to 1$ and simplifying, we get

$$\sum_{k=n}^{\infty} \left(\frac{v + k(\varrho + \lambda)\omega^{\alpha}}{v}\right)^{\mho+1} \left(\frac{v + \mu k(\varrho + \lambda)\omega^{\alpha}}{v}\right) \times (k+1) \left(\frac{\beta|\gamma|v + k(\varrho + \lambda)\omega^{\alpha}}{v}\right) a_{k+1} \leq \beta|\gamma|,$$

where $f \in A(n), \gamma \in C\backslash\{0\}, v > 0, \varrho, \omega, \lambda, \alpha \geq 0, \mho \in N_0, z \in U$.

Conversely, by applying the hypothesis (13) and letting $|z| = 1$ we get

$$\left|\frac{(\mu)z\left(D_{\lambda,v,\varrho}^{\mho+3}(\alpha,\omega)f(z)\right)' + (1-\mu)z\left(D_{\lambda,v,\varrho}^{\mho+2}(\alpha,\omega)f(z)\right)'}{(\mu)z\left(D_{\lambda,v,\varrho}^{\mho+2}(\alpha,\omega)f(z)\right)' + (1-\mu)z\left(D_{\lambda,v,\varrho}^{\mho+1}(\alpha,\omega)f(z)\right)'} - 1\right|$$

$$= \left|\frac{-\sum_{k=n}^{\infty} k(\varrho+\lambda)\omega^{\alpha}\left(\frac{v+\mu k(\varrho+\lambda)\omega^{\alpha}}{v}\right)(k+1)\left(\frac{v+k(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1} a_{k+1}z^{k+1}}{z - \sum_{k=n}^{\infty}\left(\frac{v+\mu k(\varrho+\lambda)\omega^{\alpha}}{v}\right)(k+1)\left(\frac{v+k(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1} a_{k+1}z^{k+1}}\right|$$

$$\leq \left|\frac{\beta|\gamma|\left[1 - \sum_{k=n}^{\infty}\left(\frac{v+\mu k(\varrho+\lambda)\omega^{\alpha}}{v}\right)(k+1)\left(\frac{v+k(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1} a_{k+1}\right]}{1 - \sum_{k=n}^{\infty}\left(\frac{v+\mu k(\varrho+\lambda)\omega^{\alpha}}{v}\right)(k+1)\left(\frac{v+k(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1} a_{k+1}z^{k+1}}\right| = \beta|\gamma|$$

where $f \in A(n), \gamma \in C\backslash\{0\}, v > 0, \varrho, \omega, \lambda, \alpha \geq 0, \mho \in N_0, z \in U$.

This implies that $f(z) \in S_{n,\mu}(\gamma, \alpha, \beta, \lambda, v, \varrho, \mho, \omega)$.

Lemma: 3.3

Let the function $f(z) \in A(n)$ be defined by (2), then $f(z)$ is in the class $R_{n,\mu}(\gamma, \alpha, \beta, \lambda, v, \varrho, \mho, \omega)$ if and only if

$$\sum_{k=n}^{\infty} \left(\frac{v + k(\varrho + \lambda)\omega^{\alpha}}{v}\right)^{\mho+2} \left(\frac{2v + \mu k(\varrho + \lambda)\omega^{\alpha}}{v}\right) (k+1)a_{k+1} \leq \beta|\gamma| \qquad (14)$$

where $f \in A(n), v > 0, \varrho, \omega, \lambda, \alpha \geq 0, \mho \in N_0, z \in U$.

Proof:

Same as Lemma 3.2

Theorem: 3.5

Let $f(z) \in A(n)$, then $S_{n,\mu}(\gamma, \alpha, \beta, \lambda, v, \varrho, \mho) \subset N_{n,\delta}(e)$ if

$$\delta = \frac{\beta|\gamma|}{\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)(n+1)\left(\frac{\beta|\gamma|v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)} \qquad (15)$$

where $v > 0, \varrho, \omega, \lambda, \alpha \geq 0, \mho \in N_0, z \in U$

Proof:

Let $f(z) \in S_{n,\mu}(\gamma, \alpha, \beta, \lambda, v, \varrho, \mho)$, then from (13) we get

$$\sum_{k=n}^{\infty} \left(\frac{v + k(\varrho + \lambda)\omega^{\alpha}}{v}\right)^{\mho+1} \left(\frac{v + \mu k(\varrho + \lambda)\omega^{\alpha}}{v}\right) \times (k+1) \left(\frac{\beta|\gamma|v + k(\varrho + \lambda)\omega^{\alpha}}{v}\right) a_{k+1} \leq \beta|\gamma|,$$

where $f \in A(n), \gamma \in C\backslash\{0\}, v > 0, \varrho, \omega, \lambda, \alpha \geq 0, \mho \in N_0, z \in U$ or

$$\left(\frac{v + n(\varrho + \lambda)\omega^{\alpha}}{v}\right)^{\mho+1} \left(\frac{v + \mu n(\varrho + \lambda)\omega^{\alpha}}{v}\right) (n+1) \times \left(\frac{\beta|\gamma|v + n(\varrho + \lambda)\omega^{\alpha}}{v}\right) \sum_{k=n}^{\infty} |a_{k+1}| \leq \beta|\gamma|$$

where $f \in A(n), \gamma \in C\backslash\{0\}, v > 0, \varrho, \omega, \lambda, \alpha \geq 0, \mho \in N_0, z \in U$. This implies that

$$\sum_{k=n}^{\infty} |a_{k+1}| \leq \frac{\beta|\gamma|}{\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)(n+1)\left(\frac{\beta|\gamma|v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)} \qquad (16)$$

where $f \in A(n), \gamma \in C\backslash\{0\}, v > 0, \varrho, \omega, \lambda, \alpha \geq 0, \mho \in N_0, z \in U$. By using (13) we have

$$\sum_{k=n}^{\infty} \left(\frac{v + k(\varrho + \lambda)\omega^{\alpha}}{v}\right)^{\mho+1} \left(\frac{v + \mu k(\varrho + \lambda)\omega^{\alpha}}{v}\right) \times (k+1) \left(\frac{\beta|\gamma|v + 1 - 1 + k(\varrho + \lambda)\omega^{\alpha}}{v}\right) a_{k+1} \leq \beta|\gamma|,$$

where $f \in A(n), \gamma \in C\backslash\{0\}, v > 0, \varrho, \omega, \lambda, \alpha \geq 0, \mho \in N_0$, therefore

$$\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)\sum_{k=n}^{\infty}a_{k+1}$$
$$\leq \beta|\gamma|+(1-\beta|\gamma|)\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)(n+1)\sum_{k=n}^{\infty}a_{k+1}$$
$$\leq \beta|\gamma|+(1-\beta|\gamma|)\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)(n+1)$$
$$\times\frac{\beta|\gamma|}{\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)(n+1)\left(\frac{\beta|\gamma|v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)}$$
$$\leq\frac{\beta|\gamma|\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)}{\left(\frac{v\beta|\gamma|+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)}$$

where $f\in A(n), v\neq 0, \varrho,\omega,\lambda,\alpha\geq 0, \mho\in N_0, z\in U$. Hence

$$\sum_{k=n}^{\infty}(k+1)a_{k+1}\leq\frac{\beta|\gamma|}{\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)\left(\frac{\beta|\gamma|v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)}=\delta.$$

Hence by using (1), we conclude that $f(z)\in N_{n,\delta}(e)$, this implies that

$$S_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)\in N_{n,\delta}(e)$$

Using the same technique of the proof of the Theorem 3.5, we proved the following theorem.

Theorem: 3.6

Let $f(z)\in A(n)$, then $R_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)\subset N_{n,\delta}(e)$ if

$$\delta=\frac{\beta|\gamma|}{\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{2v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)} \tag{17}$$

where $f\in A(n), v>0, \varrho,\omega,\lambda,\alpha\geq 0, \mho\in N_0, z\in U$.

Proof:

The proof for this theorem similar to that given above and we omit it.

3.3 Neighbourhood properties of $S^{T}_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)$ and $R^{T}_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)$

Theorem: 3.7

Let $g(z)\in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)$ and

$$\mathcal{T}=1-\frac{\delta\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)\left(\frac{\beta|\gamma|v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)}{\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)\left(\frac{\beta|\gamma|v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)-\beta|\gamma|} \tag{19}$$

where $f\in A(n), v>0, \varrho,\omega,\lambda,\alpha\geq 0, \mho\in N_0, z\in U$, then

$$N_{n,\delta}(g)\subset S^{\mathcal{T}}_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho).$$

Proof:

Let $f\in N_{n,\delta}(g)$, then from (3) we can write that

$$\sum_{k=n}^{\infty}(k+1)|a_{k+1}-b_{k+1}|\leq\delta$$

This implies that

$$\sum_{k=n}^{\infty}|a_{k+1}-b_{k+1}|\leq\frac{\delta}{n+1}$$

Since it is given that $g(z)\in S_{n,\mu}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)$, so from (13) we can write that

$$\sum_{k=n}^{\infty}b_{k+1}\leq\frac{\beta|\gamma|}{\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)(n+1)\left(\frac{\beta|\gamma|v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)}$$

Now

$$\left|\frac{f(z)}{g(z)}-1\right|<\frac{\sum_{k=n}^{\infty}|a_{k+1}-b_{k+1}|}{1-\sum_{k=n}^{\infty}b_{k+1}}$$

$$\leq\frac{\delta}{n+1}\cdot\frac{\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)(n+1)\left(\frac{\beta|\gamma|v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)}{\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)(n+1)\left(\frac{\beta|\gamma|v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)-\beta|\gamma|}$$

$$=\frac{\delta\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)\left(\frac{\beta|\gamma|v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)}{\left(\frac{v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)^{\mho+1}\left(\frac{v+\mu n(\varrho+\lambda)\omega^{\alpha}}{v}\right)\left(\frac{\beta|\gamma|v+n(\varrho+\lambda)\omega^{\alpha}}{v}\right)-\beta|\gamma|}$$

$$=1-\mathcal{T}.$$

This implies that $f\in S_{n,\mu}^{\mathcal{T}}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)$, therefore

$$N_{n,\delta}(g)\subset S_{n,\mu}^{\mathcal{T}}(\gamma,\alpha,\beta,\lambda,v,\varrho,\mho)$$

4. A NEW DIFFERENTIAL OPERATOR OF ANALYTIC FUNCTIONS INVOLVING BINOMIAL SERIES

Let $\mathcal{H}$ be the class of functions analytic in $\mathcal{U}:=\{z:|z|<1\}$ and $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z)=a+a_nz^n+a_{n-1}z^{n-1}+\cdots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1}$$

Let $p,h\in\mathcal{H}$ and let $\phi(r,s,t;z):\mathbb{C}^3\times\mathcal{U}\to\mathbb{C}$. If p and $\phi(p(z),zp'(z),z^2p'''(z):z)$ are univalent if p satisfies the second order superordination

$$h(z)\prec\phi(p(z),zp'(z),z^2p''(z):z) \tag{2}$$

then p is a solution of the differential superordination(2). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a subordinate if $q\prec p$ for all p satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q\prec\tilde{q}$ for all subordinants q of (2) is said to be the best subordinant. Miller and Mocanu obtained conditions on h,q and ϕ for which the following implication holds:

$$h(z)\prec\phi(p(z),zp'(z),z^2p''(z):z)\Longrightarrow q(z)\prec p(z)$$

Using the results of Miller and Mocanu, Bulboaca considered certain classes of first order differential superordinations as well as superordination-preserving integral operators. Shanmugan et al. obtained sufficient conditions for a normalized analytic function $f(z)$ to satisfy

$$q_1(z)\prec\frac{f(z)}{zf'(z)}\prec q_2(z)\ \text{ and } q_1(z)\prec\frac{z^2f'(z)}{\{f'(z)\}^2}\prec q_2(z)$$

when q_1 and q_2 are given univalent functions in $\mathcal{U}$ with $q_1(0)=1$ and $q_2(0)=1$.

For a function f in $\mathcal{A}$, and making use of the binomial series

$$(1-\lambda)^2=\sum_{j=0}^{m}\binom{m}{j}(-1)^j\lambda^j\ (m\in\mathbb{N}=\{1,2,\dots\},j\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$$

we now define the differential operator $D_{m,\lambda}^{\varsigma}f(z)$ as follows:

$$D^0f(z)=f(z) \tag{3}$$

$$D_{m,\lambda}^{1}f(z)=(1-\lambda)^mf(z)+(1-(1-\lambda)^m)zf'(z) \tag{4}$$

$$=D_{m,\lambda}f(z),\lambda>0;m\in\mathbb{N}, \tag{5}$$

$$D_{m,\lambda}^{\varsigma}f(z)=D_{m,\lambda}\left(D^{\varsigma-1}f(z)\right)\ \ (\varsigma\in\mathbb{N}) \tag{6}$$

If f is given by (1), then from (5) and (6) we see that

$$D_{m,\lambda}^{\varsigma}f(z)=z+\sum_{n=2}^{\infty}\left(1+(n-1)\sum_{j=1}^{m}\binom{m}{j}(-1)^{j+1}\lambda^j\right)^{\varsigma}a_nz^n,\varsigma\in\mathbb{N}_0 \tag{7}$$

Using the relation (7), it easily verified that

$$C_j^m(\lambda)\left(D_{m,\lambda}^{\varsigma} f(z)\right)' = D_{m,\lambda}^{\varsigma+1} f(z) - \left(1 - C_j^m(\lambda)\right) D_{m,\lambda}^{\varsigma} f(z) \quad (8)$$

where $C_j^m(\lambda) \coloneqq \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1}\lambda^j$

We observe that for $m = 1$. We obtain the differential operator $D_{1,\lambda}^{\varsigma}$ defined by A1-Oboundi and for $m = \lambda = 1$, we get Salagean differential operator D^{ς}. The main object of the present paper is to apply a method based on the differential subordination in order to derive several subordination results involving the operator $D_{m,\lambda}^{\varsigma}$. Furthermore, we obtained the previous results of Srivastava and Lashin as special cases of some of the results presented here.

Definition: 4.1

Denote by Q, the set of all functions $f(z)$ that are analytic and injective on $\bar{\mathcal{U}} - E(f)$, where

$$E(f) = \left\{\eta \in \partial\mathcal{U}: \lim_{z\to\eta} f(z) = \infty\right\}, \quad (9)$$

and are such that $f'(\eta) \neq 0 for\ \eta \in \partial\mathcal{U} - E(f)$.

Lemma: 4.2

Let the functions $p(z)$ and $q(z)$ be analytic $\mathcal{U}$ and suppose that $q(z) \neq 0$ $(z \in \mathcal{U})$ is also univalent I n $\mathcal{U}$ and that

$$\frac{zq'(z)}{q(z)} \text{ is starlike univalent in } \mathcal{U} \quad (10)$$

If $q(z)$ satisfies

$$\Re\left(1 + \frac{c_1}{\beta} q(z) + \frac{2c_2}{\beta}(q(z))^2 + \cdots + \frac{nc_n}{\beta}(q(z))^n - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0 \quad (11)$$

$$(z \in \mathcal{U}; c_0, c_1, c_2, \ldots\ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0)$$

And

$$c_0 + c_1 p(z) + c_2\left(p(z)\right)^2 + \cdots + c_n\left(p(z)\right)^n + \beta\frac{zp'(z)}{p(z)}$$

$$\prec c_0 + c_1 q(z) + c_2\left(q(z)\right)^2 + \cdots + c_n\left(q(z)\right)^n + \beta\frac{zq'(z)}{q(z)}$$

$$(z \in \mathcal{U}; c_0, c_1, c_2, \ldots\ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0)$$

Then $p(z) \prec q(z)$ $(z \in \mathcal{U})$ and q is the best dominant.

Proof:

$$\theta(\omega) \coloneqq c_0 + c_1\omega + c_2\omega^2 + \cdots + c_n\omega^n \text{ and } \phi(\omega) \coloneqq \frac{\beta}{\omega}$$

Then, we observe that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and that $\phi(\omega) \neq 0 (\omega \in \mathbb{C}^*)$

Also, by letting

$$Q(z) = zq'(z)\phi\left(q(z)\right) = \beta\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta\left(q(z)\right) + Q(z)$$

$$= c_0 + c_1 q(z) + c_2\left(q(z)\right)^2 + \cdots + c_n\left(q(z)\right)^n + \beta\frac{zq'(z)}{q(z)}$$

We find from (10) and (11), $Q(z)$ is starlike univalent in $\mathcal{U}$ and that

$$\Re\left(\frac{zh'(z)}{Q(z)}\right)$$

$$= \Re\left(1 + \frac{a_1}{\beta} q(z) + \frac{2a_2}{\beta}(q(z))^2 + \cdots + \frac{na_n}{\beta}(q(z))^n - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0$$

$$(z \in \mathcal{U}; c_0, c_1, c_2, \ldots\ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0)$$

Our result now follows by an application of Lemma 4.1

We first prove the following subordination theorem involving the operator $D_{m.n}^{\varsigma}$

Theorem: 4.3

Let the function $q(z)$ be analytic and univalent in $\mathcal{U}$ such that $q(z) \neq 0 (z \in \mathcal{U})$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathcal{U}$ and the inequality (11) holds true. Let

$$\begin{aligned}\Omega_j^m&(c_0, c_1, c_2, \dots\dots, c_n, \beta, \varsigma, \lambda, f)\\ &:= c_0 + c_1\left(\frac{D_{m.\lambda}^{\varsigma} f(z)}{z}\right) + c_2\left(\frac{D_{m.\lambda}^{\varsigma} f(z)}{z}\right)^2 + \dots + c_n\left(\frac{D_{m.\lambda}^{\varsigma} f(z)}{z}\right)^n\\ &+ \frac{\beta}{C_j^m(\lambda)}\left(\frac{D_{m.\lambda}^{\varsigma+1} f(z)}{D_{m.\lambda}^{\varsigma} f(z)} - (1 - C_j^m(\lambda)\right) \qquad (12)\end{aligned}$$

If $q(z)$ satisfies

$$\Omega_j^m(c_0, c_1, c_2, \dots\dots, c_n, \beta, \varsigma, \lambda, f) \prec c_0 + c_1 q(z) + c_2\big(q(z)\big)^2 + \dots + c_n\big(q(z)\big)^n + \beta\frac{zq'(z)}{q(z)} \qquad (13)$$

$$(z \in \mathcal{U}; c_0, c_1, c_2, \dots\dots, c_n, \beta \in \mathbb{C};\ \beta \neq 0)$$

then

$$\frac{D_{m.\lambda}^{\varsigma} f(z)}{z} \prec q(z) \ \ (z \in \mathcal{U}\backslash\{0\})$$

and q is the best dominant.

Proof:

Define the function $p(z)$ by

$$p(z) := \frac{D_{m.\lambda}^{\varsigma} f(z)}{z} \ \ (z \in \mathcal{U}\backslash\{0\}: f \in \mathcal{A})$$

Then a computation shows that

$$\frac{zp'(z)}{p(z)} = \frac{z\left(D_{m.\lambda}^{\varsigma} f(z)\right)'}{D_{m.\lambda}^{\varsigma} f(z)} - 1$$

By using the identity (8), we obtain

$$\frac{zp'(z)}{p(z)} = \frac{1}{C_j^m(\lambda)}\left(\frac{D_{m.\lambda}^{\varsigma+1} f(z)}{D_{m.\lambda}^{\varsigma} f(z)} - \left(1 - C_j^m(\lambda)\right)\right)$$

Which, in light the hypothesis (13), yields the following subordination

$$\begin{aligned}&c_0 + c_1 p(z) + c_2\big(p(z)\big)^2 + \dots + c_n\big(p(z)\big)^n + \beta\frac{zp'(z)}{p(z)}\\ &\prec c_0 + c_1 q(z) + c_2\big(q(z)\big)^2 + \dots + c_n\big(q(z)\big)^n + \beta\frac{zq'(z)}{q(z)}\end{aligned}$$

and Theorem 4.2 follows by an application of Lemma4.1.

Theorem 4.4

Let q be analytic and convex univalent in $\mathcal{U}$ such that $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathcal{U}$. Suppose also that

$$\Re\left(\frac{c_1}{\beta} q(z) + \frac{2c_2}{\beta}\big(q(z)\big)^2 + \dots + \frac{nc_n}{\beta}\big(q(z)\big)^n\right) > 0 \qquad (14)$$

$$(z \in \mathcal{U};\ c_1, c_2, \dots\dots, c_n, \beta \in \mathbb{C};\ \beta \neq 0)$$

If $f \in \mathcal{A}$

$$\frac{D_{m.\lambda}^{\varsigma} f(z)}{z} \in \mathcal{H}[q(0), 1] \cap Q$$

and $\Omega_j^m(c_0, c_1, c_2, \dots\dots, c_n, \beta, \varsigma, \lambda, f)$ defined in (12) is univalent in $\mathcal{U}$, then the following superordination:

$$c_0 + c_1 q(z) + c_2\big(q(z)\big)^2 + \dots + c_n\big(q(z)\big)^n \prec \Omega_j^m(c_0, c_1, c_2, \dots\dots, c_n, \beta, \varsigma, \lambda, f)$$

$$(z \in \mathcal{U};\ c_1, c_2, \dots\dots, c_n, \beta \in \mathbb{C};\ \beta \neq 0)$$

Implies that

$$q(z) \prec \frac{D_{m.\lambda}^{\varsigma} f(z)}{z} \ (z \in \mathcal{U}\backslash\{0\})$$

and $q(z)$ is the best subordinant.

Proof:

Let $v(\omega) = c_0 + c_1\omega + c_2\omega^2 + \cdots + c_n\omega^n$ and $\varphi(\omega) := \beta\frac{\omega'}{\omega}$

Then, we observe that $v(\omega)$ is analytic in $\mathbb{C}$, $\varphi(\omega)$ is analytic in $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and that $\varphi(\omega) \neq 0 (\omega \in \mathbb{C}^*)$. Since q is a convex univalent in $\mathcal{U}$, it follows that

$$\Re\left(\frac{v'(q(z))}{\varphi(q(z))}\right) = \Re\left(\frac{c_1}{\beta}q(z) + \frac{2c_2}{\beta}\left(q(z)\right)^2 + \cdots + \frac{nc_n}{\beta}\left(q(z)\right)^n\right) > 0$$

$$(z \in \mathcal{U};\ c_1, c_2, \ldots\ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0).$$

Theorem 4.4 follows as an application of Lemma 4.1.

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